Arithmetic intersection theory on Shimura varieties

organized by
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Workshop Summary

1. Summary

This workshop was devoted to recent advances in arithmetic intersection theory on Shimura varieties. Recent years there have been major advances on existing problems in Kudla's program, the Arithmetic Gan–Gross–Prasad program, as well as emerging new directions (function fields, p-adic intersection numbers, exceptional intersections, to name a few). The goal of the workshop was to bring a diverse range of researchers together to explore problems made accessible by these recent results, to clarify remaining obstacles and to discuss potential approaches.

Every morning there were two talks. For the first day, the two talks provide surveys and conjectures.

The speakers were:

Mon: Shouwu Zhang and Henri Darmon

Tue: Tony Feng and Jan Vonk

Wed: Yunqing Tang and Ananth Shankar

Thu: Qirui Li and Ari Shnidman

Fri: Luis Garcia and Andreas Mihatsch

2. Problem list

On Monday afternoon, there was a moderated problem session, at which about 20 open problems were proposed. The following 8 sample problems were selected for voting before the working groups on Tuesday afternoon.

(1) (Rigid meromorphic cocycles) The special values of rigid meromorphic cocycles for quadratic spaces of arbitrary signature (r, s) are conjectured to be algebraic and lie in explicit class fields. Explore the special case $(r, s) = (n, 0)$.

(2) (Totally real fields) Bruinier–Raum proved the modularity of generating series of special cycles on orthogonal Shimura varieties over $\mathbb{Q}$. Explore the possibility to extend this modularity result to totally real fields.

(3) (Function fields) The modularity of generating series of special cycles on the moduli of Shtukas is not known. Start with the case of divisors on moduli of Shtukas of low dimension.

(4) (CM cycles) Explore the modularity of generating series of CM 1-cycles arising from the proof of the arithmetic fundamental lemma. As a first step, relate the degree of

the generic fiber of CM 1-cycles on $U(n-1,1)$ Shimura varieties to theta functions associated to $u(n)$.

(5) (Divisors on $\mathcal{A}_3$ or $\mathcal{A}_4$) For an abelian fourfold over a number field, is its reduction isogenous to a Jacobian modulo infinitely many primes? Similarly, for an abelian threefold over a number field, is its reduction isogenous to a hyperelliptic Jacobian modulo infinitely many primes? Explore the relation of these questions to the arithmetic intersection of divisors and horizontal 1-cycles on $\mathcal{A}_4$ and $\mathcal{A}_3$ respectively.

(6) (Green currents) Study the behavior of Green currents along the boundary of $U(n,1)$ Shimura varieties. Generalize Ehlen-Sankaran's result on the difference of Green currents to higher codimension. Is there an analogue of archimedean local arithmetic Siegel-Weil formula for $U(p,q)$ $(p,q>1)$? Is there an archimedean analogue of arithmetic fundamental lemma in the context of arithmetic GGP?

(7) (Mixed characteristic Shtukas) Can one define local intersection numbers using moduli of mixed characteristic Shtukas (without integral models)?

(8) (Higher GGP) Study the higher analogue of GGP for $U(2)\times U(3)$ over function fields using purely global method after Yun-Zhang.

3. Working group activities

In the afternoons of Tuesday through Friday, five to six groups met each day that roughly corresponded to following topics.

3.1 Rigid meromorphic cycles. This project grew out of the talks of Darmon and Vonk given in the first two days of the workshop, whose goal was to propose a framework for explicit class field theory going beyond the setting of CM fields. This framework proposes to construct class invariants in abelian extensions of certain – non CM, in general – reflex fields from the *special values* of *rigid meromorphic cocycles* on the orthogonal group G of a quadratic space V over $\mathbb{Z}$ of real signature (r,s).

In the setting of orthogonal groups of signature $(2,1)$, there is a growing body of experimental and theoretical evidence arising from the work of Darmon, Pozzi and Vonk, some of which was reported on in Jan's lecture. There is virtually no further support of any kind, either theoretical or experimental, for the more general framework, which grew out of exchanges with Lennart Gehrmann and Mike Lipnowski over the fall preceding the workshop. One of the themes of our discussions was to try to subject the conjecture to some much needed stress tests and sanity checks. A subtheme was the hope that one might apply some of the rich geometric theory of orthogonal Shimura varieties to the study of this conjecture, which was presented, in Darmon's talk, as a framework for "arithmetic intersection theory without Shimura varieties", a description which is apt for the time being but is perhaps just a reflection of our current state of ignorance.

Rigid meromorphic cocycles are classes in

$$H^s(\Gamma, \mathcal{M}^\times(X_p)),$$

where

(1) $\Gamma = G(\mathbb{Z}[1/p])$.

(2) X_p is a rigid analytic space whose $\mathbb{C}_p$-points are

$$X_p = \{v \in V \otimes \mathbb{C}_p \text{ with } \langle v, v\rangle = 0 \text{ and } v^\perp \cap V_{\mathbb{Q}_p} \text{ anisotropic}\}/\mathbb{C}_p^\times.$$

(3) $\mathcal{M}^\times(X_p)$ is the multiplicative group of rigid meromorphic functions on X_p (with "rational quadratic divisor").

Special points of X_p are points that are fixed by a torus in $G(\mathbb{Q})$ of maximal real rank s; their stabilisers in Γ are then abelian groups of rank s.

The conjecture is that the value of a rigid meromorphic cocycle J with rational quadratic divisor at a special point x belongs to a class field of a reflex field attached to x. (This value, denoted $J[x]$, being as defined in the slides of Darmon's lecture.)

The special case where $s = 0$ seems like it might fall under the purview of the classical theory of complex multiplication, because of the following features:

(1) The group Γ acts discretely on X_p, so the quotient $\Gamma\backslash X_p$ is a nice rigid analytic space.
(2) A rigid meromorphic cocycle is just a class in $H^0(\Gamma, \mathcal{M}^\times(X_p))$, hence, a meromorphic function on this quotient.
(3) Special points on X_p are fixed by tori arising from the elements of a CM extension of relative norm one (to the totally real subfield). The conjecture predicts that these values should be defined over abelian extensions of suitable reflex fields, as is typical for the fields of definition of the CM locus on Shimura varieties.

The case of signature $(n, 0)$ thus provides an natural first testing ground for the conjectures that were perpetrated during Monday's lecture.

The first natural question that arose was to relate the symmetric space X_p to the supersingular locus on an orthogonal Shimura variety Sh of signature $(n-2, 2)$. It is immediately apparent – thanks to a tantalising "exercise for the reader" in a key article of Howard and Pappas, which the participants dutifully worked out during the week – that X_p is contained in the target of the Grothendieck-Messing period map from the rigid fiber of a Rapoport-Zink space that uniformises the supersingular locus of Sh.

Much of our discussions of the week revolved around the desire to give a natural algebro-geometric interpretation of X_p as a subset of the period space. Some of the geometric insights that emerged were:

- The supersingular points whose periods lie in X_p seem to be precisely those that reduce to the generic (top-dimensional) Ekedahl–Oort stratum in the special fiber (and to no lower-dimensional stratum). (This remains to be checked carefully and be made precise.)
- It would appear that the elements of X_p are precisely the periods in $\mathcal{Q}^{\mathrm{wa}}$ whose fiber relative to the period map are the most simply described, and are perhaps finite. (This is because the fiber at x seems to correspond, *sauf erreur*, to maximal $\mathbb{Z}_p$-lattices in the orthogonal complement of x in $V_{\mathbb{Q}_p}$, which for $x \in X_p$ is anisotropic.)

Hence the hope that the quotient $\Gamma\backslash X_p$ might uniformise a part of the supersingular locus of the orthogonal Shimura variety, in a way that is appealingly concrete and would lend itself to explicit numerical calculations.

The perpetrators of the conjecture of Monday (Gehrmann, Lipnowski, and Darmon) conclude by expressing the hope that "what happens in San Jose (albeit virtually) stays in San Jose". Indeed, until the promising picture that emerged in this week's discussion can be further fleshed out and tested, these conjectures must be taken with a serious grain of salt! Putting the conjecture on a firmer footing in the case of signature $(n, 0)$ would be a key step.

3.2 Modularity of generating series over totally real fields. During the workshop, we have considered extending the modularity of a generating series of special cycles with values in the Chow group for a Shimura variety corresponding to an orthogonal group over a totally real field. This has been done conditionally by Yuan-Zhang-Zhang, up to a convergence condition. The result has been extended by Bruinier and Raum over $\mathbb{Q}$, who have proved the result unconditionally. The problem reduced to showing the space of formal Fourier-Jacobi series to the space of Siegel modular forms. We have discussed with Jan Bruinier and he, together with collaborators, is currently extending the result using geometric methods.

3.3 Modularity of generating series of cycles over function fields Following the formulation of generating series in the forthcoming work of Feng–Yun–Zhang, we hope to work out some baby examples of the modularity of their generating series. Let $G = U(n)$. Roughly speaking, we consider generating series with a-th coefficient being the cycle $Z_{\mathcal{E}}(a)$ for some $a \in \mathcal{A}_{\mathcal{E}}$ (which give the Hermitian forms on $\mathcal{E}$), where $Z_{\mathcal{E}}(a)$ is viewed as a function $Z_{\mathcal{E}}(a)$ on $\mathrm{Bun}_G(k)$ by pushing-forward the constant function. More precisely, we take a Hermitian bundle $\mathcal{H}$ of rank $2m$ satisfying certain conditions w.r.t. $\mathcal{E}$, we consider the sum

$$f_{\mathcal{E}\subset\mathcal{H}} := q^{n/2\times\deg\mathcal{E}} \sum_a \psi\langle a, [\mathcal{H}]\rangle 1_{Z_{\mathcal{E}}(a)}$$

where $[\mathcal{H}]$ is the extension class in $\mathrm{Ext}^1(\sigma^*\mathcal{E}^\vee, \mathcal{E})$. This is a function on $\mathrm{Bun}_G(k)$. The modularity conjecture in this setting predicts that the function $f_{\mathcal{E}\subset\mathcal{H}}$ only depends on $\mathcal{H}$, not on $\mathcal{E}$. (For $r = 0$ the conjecture is a fact.)

For Shtukas with r legs, we consider analogously $Z^r_{\mathcal{E}}(a)$ which is finite over Sht^r_G. We expect to define a cycle $[Z^r_{\mathcal{E}}(a)] \in \mathrm{Ch}_{(n-m)r}(\mathrm{Sht}^r_G)$ where Sht^r_G has dimension nr. Likewise we can start with a rank $2m$ Hermitian bundle $\mathcal{H}$ and consider the sum

$$f_{\mathcal{E}\subset\mathcal{H}} := q^? \sum_a \psi\langle a, [\mathcal{H}]\rangle [Z^r_{\mathcal{E}}(a)] \in \mathrm{Ch}_{(n-m)r}(\mathrm{Sht}^r_G).$$

The term $q^?$ here should be analogous to the term $q^{n/2\times\deg\mathcal{E}}$ in the case with no legs, although we are not sure what the exponent should be at the moment.

The conjecture in this case again predicts that $f_{\mathcal{E}\subset\mathcal{H}}$ depends only on $\mathcal{H}$ and not on $\mathcal{E}$. In the special case where $m = 1$ and $\mathcal{E}$ is a line bundle L on X' (a double covering of X), one only needs to worry about the 0-th coefficient $[Z^r_{\mathcal{E}}(0)] \in \mathrm{Ch}_{(n-1)r}(\mathrm{Sht}^r_G)$. If $a \neq 0$, then $Z^r_{\mathcal{E}}(a)$ has expected dimension $(n-1)r$.

During the workshop, We considered Shtukas of one leg over $\mathbb{P}^1$ with some $\Gamma_0(\infty)$-level structures, while examining the explicit computations done by Maria Ines de Frutos-Fernandez. In this case, the moduli space is 2-dimensional. Each cycle in the generating series is 1-dimensional. Our method to the modularity problem is to intersect it with other divisors. In fact, the point of adding level structures is that we discovered that the case of 2 legs without level structure turns out to trivial. The corresponding modularity statement says $0 = 0$.

3.4 Divisors on $\mathcal{A}_3$ and $\mathcal{A}_4$ We arrive at formulating the following general conjecture. **Conjecture**. Let S over $\mathrm{Spec}\mathbb{Z}$ (of course, $\mathbb{Z}$ can be replaced by the ring of integers of any number fields with finitely many primes inverted) denote a Shimura variety. Let $X, Y \subset S$ denote flat subschemes which have complementary dimensions. Then, the intersection

$$X \cap \left(\bigcup_{\tau \text{ Hecke}} \tau(Y) \right)$$

should be Zariski dense in X, where τ varies over the set of all Hecke correspondences on S.

The conjecture can also be formulated in the geometric setting, both in characteristic zero and characteristic p (with suitable ordinariness hypotheses). Here are some known cases of this conjecture:

- In the characteristic-zero geometric setting, the conjecture follows in full generality by forthcoming work of Tayou–Tholozan.
- When the ambient Shimura variety is a product of modular curves, then the theorem follows from Charles' result.
- When the ambient Shimura variety is a GSpin Shimura variety (or a $U(n, 1)$ Shimura variety) and Y is a Kudla-Rapoport special cycle, then the conjecture follows from Shankar–Shankar–Tang–Tayou.
- In the characteristic p setting where the ambient Shimura variety is GSpin Shimura variety and Y is a Kudla-Rapoport special cycle and X is generically ordinary, the conjecture follows from Maulik–Shankar–Tang.

3.5 Modularity of CM cycles The original question was formulated for unitary Shimura varieties. Define a CM-cycle $\mathrm{CM}(m)$ on it by parameterizing an additional endomorphism x with $x^* = -x$ and $\mathrm{tr}(xx^*) = m$. (Also throw in an elliptic condition to ensure that x really generates a maximal degree CM-extension.) Then the question was if $\sum_m \mathrm{CM}(m)q^m$ is modular.

The answer to the question is yes, at least after taking degree. Group-theoretically, the situation corresponds to U acting on Lie$U$, the latter endowed with its Killing form. This gives rise to a map $U \to O(\mathrm{Lie}U)$ and the generating series from above turns out to be the pull back of a Kudla–Millson generating series. In fact, this works for any group G, not just for unitary groups. Further questions were raised (and left unanswered):

- The pull back of Kudla–Millson series can also be done without the elliptic condition. What interpretations do the cycles and degrees have on $\mathrm{Sh}(U)$?
- How to define an arithmetic version of $\mathrm{CM}(m)$? How to prove its modularity?