RESEARCH ON INQUIRY BASED LEARNING IN UNDERGRADUATE REAL ANALYSIS

organized by
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Workshop Summary

Workshop Goals

The teaching and learning of undergraduate mathematics has received increased attention in recent years. Multiple growing communities of professionals have re-envisioned how core proof courses such as real analysis might best be taught. The desire to give students responsibility for discovering key course content concepts and the opportunity to engage in authentic mathematical research at their level, generally referred to as inquiry-based learning (IBL), unifies many of these efforts. The accumulated expertise across these communities provides a rich opportunity to initiate coordinated course-specific research projects on inquiry based teaching and learning.

This workshop convened experts in education research, curriculum development, instruction, faculty development, and assessment in IBL undergraduate real analysis. We surveyed the state of multiple perspectives on the field and existing connections across these areas of expertise. We identified and framed future collaborations to refine research-based studies and develop a research agenda responsive to existing needs regarding IBL practice. The primary focus of the workshop was to refine pertinent, tractable research questions and design consequent high-quality studies to address these questions. We identified four areas in which additional research efforts are needed to support IBL instruction in undergraduate real analysis. The workshop resulted in collaborative teams with diverse expertise to pursue necessary resources and tackle the critical research questions identified in the workshop.

Schedule

Monday morning

> Introductions
> Presentation of Survey data (Michael Oehrtman)
> Talk: Overview of IBL Real Analysis (Ted Mahavier)

Monday afternoon

> Talk: Overview of relevant math ed research (Paul Dawkins)
> Ask the expert: Panelists Michael Oehrtman (math ed), David Clark (IBL Analysis)

Tuesday morning

> Talk: In-depth research overview on proof (Paul Dawkins)
> Talk: In-depth research overview on content in Real Analysis (Mike)

Tuesday afternoon

> Problem session: Generate research questions
> Ask the expert: Questions about research (Michael Oehrtman, Paul Dawkins)

Wednesday morning

Talk: Steve Moric

Wednesday afternoon

Groups work on research questions: Institutional factors & instructor development, Instructor choices, Persistence & identity, Proof

Thursday morning

Talk: Presentation of research questions and feedback to promote refinement.

Talk: Assessment in IBL Analysis (Robert Campbell)

Thursday afternoon

Groups work on final research question: Instructor choices, Persistence & identity, Problem sequences & learning trajectories, Proof

Group reports and workshop feedback

Friday morning

Groups finalize research questions: Instructor choices, Persistence & identity, Problem sequences & learning trajectories, Proof

Friday afternoon

Group reports and workshop final versions

Research Questions

Instructor choices.

Establishing a productive classroom culture.

Research Questions:

(1) What strategies do IBL instructors use to create and sustain an appropriate classroom culture in IBL real analysis courses?
(2) What are the essential aspects of appropriate classroom culture that faculty make explicit moves to establish and maintain, regardless of classroom structure or strategy for doing so?

Abstract: Because IBL instruction departs from the traditional (lecture-style) mathematics classroom contract between teacher and students (and among peers), we anticipate that all IBL instructors use various strategies for creating and maintaining a classroom culture conducive to the expectations and goals of IBL pedagogy. While discussion and self-report of experienced IBL instructors suggests that these strategies are diverse, we hypothesize that the diverse strategies are used to accomplish a more universal set of goals. We propose a study of these teaching moves, their purposes, and their efficacy through qualitative observation of IBL classrooms. We anticipate doing so by identifying key points in which faculty make explicit moves to create and sustain a desirable classroom culture before conducting faculty and student interviews around video clips of such events. Eliciting student interpretation of such events and how they arise to breaches of the intended classroom expectations, we anticipate being able to learn about students understanding and adoption of these expectations for learning.

- Video record early classes and then a sampling later in the semester.
- Attend to initial solicitation for buy in to IBL and then to breaches of classroom contract that are negotiated.

- Conduct member check interviews for instructor and student interpretations of classroom expectations and the justification for/sources of such expectations. It was proposed to measure student perceptions twice per semester, possibly after 4 and 12 weeks (based upon IBL instructors experiences of the recurrent progression of students affective responses in such courses).
- Possible student questioning strategy: If you could choose how classroom time is spent in the mathematics course you are taking, what percentage of the time would you spend on: professor lecture, student presentations, group work on challenging problems, etc?
- We assume many students will come into an IBL real analysis course with a baseline set of expectations informed by traditional mathematics instruction in which students attempt tasks after receiving relevant direct instruction, students are assessed on private work without publishing student work to other students, and the professor and textbook are the primary sources of new mathematics in the class. IBL is taken to entail the expectation that students attempt proof tasks for which they have not received direct instruction, students present their own work to the class for evaluation, and that students author new mathematical content into the classroom.
- Relevant categories we anticipate may arise in the study:
    - Legitimizing failure (or linking success to hard work and repeated attempts)
    - Soliciting buy in to the alternative classroom expectations
    - Negotiating classroom expectations (of students and teacher)
    - Endorsing standards for acceptable proof
    - Curtailing undesirable mathematical practices
- We anticipate observing IBL instructors of various levels of experience. Novice instructors might provide more opportunity to identify breaches and difficulties that require intervention. Observing experienced IBL instructors will aide in identifying successful strategies, since documenting and publishing such strategies is an important research product.

Learning through proof presentations. The overarching goal of this investigation is to understand what students learn in IBL courses when listening to peer proof presentations. Two relevant hypotheses were identified:

Hypothesis 1: Students learn more from proof presentations in an IBL course because they have already attempted the proof tasks before seeing the completed solution.

Hypothesis 2: Students learn more from proof presentations in an IBL course because they pay closer attention to find and identify mistakes, which may depend upon how student listening is guided by the instructor.

We propose two possibly independent studies for these two hypotheses.

Research Questions:

(1) How does students having attempted a proof production influence their learning from a proof presentation?
(2) How do students ongoing classroom experiences (in lecture or IBL classrooms) influence their learning from proof production and observation of a proof presentation?

Abstract: IBL instructors anticipate that students learn from their peers proof presentations in part because they have already attempted the task themselves. To investigate this, we propose that two groups of students attempt to (a) produce a proof of a claim and (b) view a presentation of a proof of that claim, in alternate orders. The condition of proof production before presentation mimics the IBL listening environment. After both learning experiences, Mejia-Ramos et al.s proof comprehension instrument will be administered to assess various dimensions of student learning about the proof. We further propose that students currently enrolled in both IBL and lecture-style courses participate in the study. This will make the study sensitive to the possibility that students ongoing learning practice may determine their optimal learning conditions (rather than the learning conditions themselves).

- Alternative possible conditions: proof presentation by an experienced instructor or proof presentation by a student (with or without mistakes).
- The study design may need to attend to how students time listening to their peers proof presentations is structured by the instructor (e.g. by assigning roles, by providing rubrics, by inviting peer-peer feedback).
- We anticipate that the choice of mathematical topic, proof task, and that tasks relative difficulty will be crucial to the outcome and viability of the study.

Research Question: How does student learning from a proof presentation differ between presentations made by expert mathematics faculty and presentations made by other students?

Abstract: Students are the primary authors of mathematical proofs in IBL courses, meaning that students will more often see imperfect proofs rather than valid proofs that exhibit standard mathematical form. Observing peer proofs may invite more active listenership since students are expected to question and give feedback on peer proofs. Observing expert proofs may improve learning because they serve as models of appropriate proof writing. We propose a study that compares students proof comprehension, as measured by Mejia-Ramos et al.s instrument, after viewing the two types of proof presentations. We anticipate that conducting the study with students from both IBL and lecture-style courses will benefit the study. It will be important to distinguish the comparative benefits of presentation conditions from the possibility that students simply learn how to listen effectively within their native learning environment.

- This study was conceptualized with reference to similar studies in physics education and elsewhere where listening to peers produced greater gains in comprehension over a short time interval relative to some measures of learning.
- We anticipate that the 2x2 design coupled with the multi-dimensional measure of proof comprehension will provide a rich set of possible outcomes leading to different inferences about the nature of student learning from listening.

Persistence & identity.
Exploratory study of student development.

Research Questions:

- What are the student developmental categories in a Moore Method course?
- How can we refine and explain these categories?
- How can we explain and describe student advancement between developmental categories?

Abstract: Many Moore Method[1] instructors report seeing dramatic transformations experienced by students when they make significant steps forward in their levels of mathematical achievement. Through qualitative observation of several Moore Method classrooms, we anticipate identifying and refining definitions of student developmental categories. We anticipate this study will develop baselines for subsequent studies to document transformations to higher developmental categories in Moore Method classes, record them in detail, and investigate the conditions that foster such transformations.

We propose a preliminary list of developmental categories as follows:

Beginners:: Students who have not been able do any presentations successfully.

Novices:: Students who successfully present a proof that is a follow your nose type proof (they know what a definition is and definitions are and the logic and format of a proof are and can put them together).

Apprentices:: Students who successfully present a proof that requires some significant insight or ingenuity to answer.

Masters:: Students who initiate interest in mathematical problems that they want to do themselves.

Note that these levels somewhat parallel Lee May's levels of performance, but differ in a practical way. Assessing a student for May's levels require a carefully designed assessment procedure in addition to the classwork itself. In contrast, instructors can classify students according to the levels proposed here based only on what they see from the student's class performance.

Data Sources: Ted Mahavier has video we might use for exploratory studies of his and his father's courses. Anneliese Spaeth and Padraig McLoughlin have volunteered to use their Analysis courses and possibly record them in Spring 2016. Ted Mahavier is on sabbatical during Spring 2016, and has volunteered some time. Ted Mahavier is teaching Real Analysis in Fall 2016, and has volunteered his course for participation in this study. We invite others to participate by contributing observations of these transformations in their own Moore Method classes.

Case study of the impact of IBL on student development.

Research Questions: How does taking a Moore Method course affect student:

- Confidence: Willingness to engage and belief in eventual success
- Independence: Prove theorems and solve problems (and verify on their own)
- Identity: Participation in mathematical culture
- Willingness to try and fail (and see its worth)
- Resilience: Willingness to try after failure
- Perception of self worth/worth of their work
- Locus of control
- Perception of the nature of mathematics

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[1]By Moore Method, we mean an Inquiry Based Learning (IBL) course centered around individual student presentations.

Abstract: The Colorado Study (2014)[2] established that students from IBL courses attain greater success in subsequent math courses than students from more traditional, lecture-based courses. In an effort to explain in more depth why this might be the case, we propose a study tracking any of several student attributes identified by mathematicians as important for student success in mathematics. Above we have begun a preliminary description of these attributes. We propose a series of case studies of students making transitions from one category (Beginner, Novice, Apprentice, Master) to another, including cases of failure to transition.

Benefits of MMM over Lecture for Strong Students.

Research Questions:

(1) What are long term effects of MMM instruction upon strong students?
(2) How do these effects compare with those of more traditional, lecture-based instruction (and possibly more general IBL)?
(3) How can we further describe student development within the strong/high-achieving category? (see Tranformation Study)
(4) In what ways do MMM courses support this development, specifically for strong students?

Abstract: The following statement appears in the final report of the Colorado Study, Chapter 9: Summary of Findings[3]

> Overall, it appeared that non-IBL courses tended to reinforce prior achievement patterns, helping the "rich" to get "richer." In contrast, IBL courses seemed to offer an extra boost to lower achieving students, especially among pre-service teachers. Yet there was no evidence of harm done to the strongest students. Indeed, high-achieving students may be encouraged by an IBL experience to take more mathematics courses, especially more IBL courses (6.6.2) again, consistent with instructor observations that strong students found the IBL approach stimulating (8.2.5).

The study did recognize and record different forms of IBL, primarily individual presentation and group work. However, it went on to aggregate them under the label "IBL," in order to provide enough data to draw statistically significant conclusions.

We applaud the findings concerning lower achieving students[4]. However, we also recognize that the strongest students of today will be the leaders of tomorrow, and that it is important to assist them in achieving their full potential. Doing them "no harm" does not seem likely to achieve that. We would like to investigate the possible benefits of IBL to these students mentioned in the above quote from the Colorado Study. The idea of this study is that these benefits might be more pronounced and easier to establish if it is focused on the benefits of the MMM, rather than IBL in general.

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[2]Laursen, S. L., Hassi, M.-L., Kogan, M., & Weston, T. J. (2014). Benefits for women and men of inquiry-based learning in college mathematics: A Multi-institution study. *Journal for Research in Mathematics Education*, 45(4), 406418.

[3]S. Laursen, M. Hassi, M. Kogan, A. Hunter, Evaluation of the IBL Mathematics Project, Assessment & Evaluation Center for IBL in Mathematics, Ethnography and Evaluation Research and Tim Weston ATLAS Assessment and Research Center, University of Colorado at Boulder (2011), p169.

[4]Laursen, et al., divided students into Low achieving (Math GPA 2.5 or below), High achieving (Math GPA 2.5 or above) categories.

The purpose of this study is to determine if there is validity to the commonly held belief by MMM instructors that, among students previously identified as strongest, MMM students tend subsequently to fulfill more of their potential than those taking the same courses in a traditional, lecture-based setting. To that end, we wish to more carefully examine the benefits of Modified Moore Method courses to strong students. By "strong" students we mean those students that are either hard-working and diligent, or talented and smart, or both. Below we propose some possible proxy measures for these traits.

Thus far we have put together two possible approaches:

(1) We wish to compare the effect of MMM vs. lecture-based instruction upon strong students.
(2) We wish to examine high-achieving students (after becoming informed by our previous transformation study), and accomplish a similar goal of describing developmental categories at a higher resolution for this group of students.

In order to investigate this pair of research questions, it will first be necessary to determine the following:

(1) How are strong incoming freshmen best identified? (Perhaps by high school grades? SAT grades? Other?)
(2) How is realization of potential by college graduates who declared a mathematics major best measured? (Perhaps by completion of a mathematics degree after declaring the major? By taking mathematics courses or doing individualized work in mathematics beyond the requirements for the degree? By acceptance to graduate school? By individual testimony? Other?) There are many ways achievement of potential could be manifest. The study will rely on identifying ways that are practical to measure.

Once satisfactory metrics have been identified and described, the study may be carried out in the following steps:

(1) Institutions will be identified which have mathematics courses that are taught sometimes by MMM and sometimes in a traditional, lecture-based format.
(2) For those institutions, records will be gathered of recent graduates who had at some point declared a mathematics major.
(3) From those records, students who qualify as "strong" will be selected.
(4) From the selected students, two groups will be formed. One group will consist of those who had taken the most MMM courses, and the other will consist of those who had taken the most lecture-based courses.
(5) Data on realization of potential will then be analyzed and used to compare MMM-students with lecture-based-students.

The investigation of research questions 3 and 4 above will be informed by our planned "Exploratory Study of Student Development Under the Modified Moore Method". That study will describe student developmental categories and describe how students advance through these categories. We wish to build on that study by examining the highest-achieving

category in greater resolution. That is, how can we explain and describe student development in a MMM course within the highest-achieving category?

Problem sequences & learning trajectories.
Intellectual cross-training.

Research Question: When interweaving content areas within a problem sequence, what are the impacts on students success, conceptual development, relationships between related concepts, their images of the roles of definitions and theorems, and equivalence of approaches?

Abstract: IBL Real Analysis instructors often interweave problems on different concepts rather than treating each concept independently. We propose to study whether doing so increases the odds of success for individual students and, if so, why. Perhaps having problems spanning multiple concepts on which to work each evening allows students to make connections between these concepts thereby increasing the students' understanding of each concept as well as the interplay between them. What is an optimal amount of interweaving? We illustrate the phenomena we wish to study via two examples.

(1) An instructor might alternate, within a problem sequence, problems related to limit points and problems related to sequences prior to stating a problem that ties the two concepts together, such as every infinite bounded sequence has a limit point.

(2) An instructor might have problems intertwined between continuity and differentiability, with the concluding problem being to prove that every differentiable function is continuous.

Strategic Walls.

Research Question: How does an instructor create an adequate number of conjectures that will effectively demonstrate students need for mathematical need for proof in order to verify or invalidate their intuition? What are the features and timing of these conjectures for optimal effectiveness?

Abstract: Pedagogical conjecturing activities are intended to both develop students intuitions about the content in real analysis and reinforce students understanding of the need for proof or counterexample. Real analysis is a particularly appropriate site for this question since students start with significant intuitions (from calculus) that is not necessarily consistent with the foundations of analysis (completeness, monsters, etc.). Specific types of these activities may include

Traps:: contradicts standard obvious intuitions
Hooks:: build new (perhaps surprising) intuitions, e.g., the cantor set
Generalizing:: asking whether a recently proven statement holds for a broader class of tasks

What does implementing such strategies communicate to students about

(1) correct proof?
(2) their own responsibility for their learning?
(3) the instructor's responsibility for their learning?
(4) mathematical authority?
(5) what it means to "do math?"
(6) potentially, conceptual understanding?

(7) workload?
(8) the difficulty of the concepts?
(9) the amount of coverage accomplished by the course?

In an extreme version of this question, we seek to understand the differences in the development of students habits of mind when approaching mathematical tasks if the course is structured as statements, with many (1/4? 1/2?) are false.

Beyond proof.

Research Question: What are the impacts of incorporating mathematical activity such as defining or conjecturing at different points or with different frequency in an IBL course?

Abstract: The classic Moore Method course involves the distribution of a sequence of definitions and problems students are charged with solving. Other IBL strategies may ask students to generate the definitions and problems themselves. The choice of strategies on this continuum may depend on the instructors goal for the course. For example, requiring students to participate in the defining and conjecturing tasks may result in a different conception of the nature of mathematical inquiry and the culture of mathematics. On the other hand, it may be the case that requiring students to generate every definition in a course may discourage students or require an undesirable amount of time. We wish to study the optimal frequency, locations, and depth of these experiences, both defining and conjecturing. Many effects on student perception would be of interest, including perception of: student and instructor responsibility for learning, mathematical authority, what it means to do math, their understanding, course difficulty, course coverage.

Designing in the Zone of Proximal Development.

Research Question: When teaching from a set of notes, how does an experienced IBL instructor identify when a problem sequence needs to be reconstructed to support the particular learning needs of their students, how do they accomplish it, and how do they assess its effectiveness?

Abstract: Course notes are more static than actual problem sequences implemented by IBL instructors. Effective instructors often modify problems in response to the ways that particular classes develop. For example a particular problem may be too difficult as stated and need to be broken into sub-steps to allow student success. Alternately, an instructor may realize that some of their students may be better served by using an alternate definition to a particular term, thus changing the trajectory of problems (e.g., the old definition now becomes a theorem). We seek to understand what aspects of students understanding an instructor must attend to in order effectively identify the need for these modifications and to understand the design principles that allow them to successfully modify the sequence. We also seek to understand any aspects of students or topics that may affect this process such as differences in when working with content for which students have a rich concept image vs. when a concept image is relatively absent. In addition to studying how this process

unfolds during a course, we may also explore how the standard versions of course notes may be modified outside of a particular implementation.

Proof.
How does an IBL class impact students' understanding of proof?.

Research Questions: In what ways does an IBL class[5]:

(1) impact students' proof schemes?
(2) affect students' understanding of the contextual meanings in mathematics of the words "axioms, definitions, conjecture, theorem" and their role in mathematics?
(3) help students develop competence at proof?

What features of IBL create these impacts?

Abstract: Harel & Sowder have proposed a framework to classify and examine students' proof schemes. They have looked mostly at students in traditional courses, or as part of teaching experiments. We conjecture that in those implementations of IBL that provide experiences to students in which they must prove propositions on their own, students' proof schemes will improve, as will their understanding of proof, and their competence at writing proofs.

We propose a qualitative study (a *case study* possibly) of a Real Analysis IBL course. Data will include documentation of the IBL implementation (syllabus, problem sequence, classroom observations, instructor's interview), evidence of students' procedural and conceptual understanding of proof (artifacts such as proofs worked in tests or homework assignments, clinical interviews, pre- and post-surveys).

Things to think about:

- If the study is done in two sites (e.g., Texas and India), how to ensure that classrooms observations are reliable and consistent? (Ideas: video, Skype interviews.)
- Why Real Analysis? It could be done in other courses, but RA is a good one: students have some experience with proofs, they are more mature mathematically, course content is challenging enough.
- Data should include students' demographic information, previous IBL experience, academic perfomance prior to the course.
- Teasing out what features of IBL cause change may be more difficult than gathering evidence of such change.
- We need to consider follow up evidence of the robustness of change or impact.
- Data collection instruments (interview protocols, proof tasks, etc.) will be piloted both in IBL and non-IBL courses, as possible.
- A more comprehensive literature review should be conducted to inform data collection and data analysis strategies, as well as useful frameworks to use or to adapt for this study.

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[5]A more precise description of *IBL class* will be done prior to the study. We are choosing not to restrict the statement of our questions at this point.