Hypergraph Turan problem

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Workshop Summary

Overview

This workshop, sponsored by AIM and NSF, focused on the hypergraph Turán problem and related questions. The *Turán function* $\mathrm{ex}(n, \mathcal{F})$ of an r-graph family $\mathcal{F}$ is the maximum size of an r-graph on n vertices that does not contain any member of $\mathcal{F}$ as a subgraph. The *Turán density* of $\mathcal{F}$ is $\pi(\mathcal{F}) = \lim_{n\to\infty} \mathrm{ex}(n, \mathcal{F})/\binom{n}{r}$. This problem was introduced in the seminal paper of Turán back in 1941. It is a fundamental and important question of extremal combinatorics that is still wide open in general in spite of 70 years of research by numerous mathematicians. For example, the \$1000 prize of Erdős for computing the Turán function of K_4^3, the complete 3-graph of order 4, is still unclaimed. The notorious difficulty of the Turán problem has not stifled research. On the contrary, it brought to life many fruitful areas of modern combinatorics. Some very exciting recent developments (hypergraph regularity and stability, flag algebras) were greatly motivated by Turán-type questions.

The purpose of this workshop was to consolidate the current knowledge on the hypergraph Turán problem, understand better the recent developments and new techniques, collect a list of open problems that will serve as a guide for future research, encourage collaboration, and attract new peoples (including many very promising young mathematicians) to this area.

There were two introductory lectures (by Dhruv Mubayi and Oleg Pikhurko) that concentrated on some open problems so that all participants (even those new to the area) could start working on some problem already on Tuesday afternoon. Peter Keevash presented three lectures explaining flag algebras and some possible ways of applying them to the Turán function and other extremal problems. Jacob Fox presented his new proof of the Graph Removal Lemma that is a very useful tool in extremal combinatorics. His proof gives better bounds (which is still a tower but its height is now not polynomial but logarithmic in the input constant $\varepsilon > 0$). David Conlon presented his recent results that relate the the hypergraph Turán problem to the problem of avoiding a given subgraph F inside a hypercube Q_n.

A preliminary list of open question was compiled and distributed before the workshop. It was considerably expanded after Monday's open problem session. The list should be available from the AIM website and will hopefully serve as a useful reference for researchers interested in this area.

Seven open problem were identified for group work, They spanned a range of topics from dense to sparse case, from a single forbidden subgraph to infinite families, dealing with the Turán function alone or with its relations to hypergraph degeneracy or independence number.

The following partial results have been obtained.

Forbidden Hypergraphs Related to K_{k+1}^k

For a fixed k, the graph F_i is a k-graph on $k-1+i$ vertices where the vertices are split into two sets S and T with $|S|=i$ and $|T|=k-1$ with the edges being all k sets that contain all of S or all of T. The k-graph F_2 is a clique and determining its Turán function is a famous open question. On the other hand, F_k was shown to have Turán density $1-\Theta\big(\frac{\log k}{k}\big)$ by Bohman, Frieze, Mubayi, Pikhurko (2010). At the workshop we looked at what happened in between these two ends, i.e., what is the Turán density for F_i for $3 \le i \le k-1$. This question was proposed by Dhruv Mubayi.

For i fixed and k growing, we were able to give a generalization of the construction of Kim and Roush (1983) to show that

$$1-\frac{c\log k}{k^i} \le \pi(F_{i+1}), \pi(F_{k-i}) \le 1-\frac{d}{k^i}$$

for constants c and d. The construction works by associating edges with subsets of $\mathbb{Z}_\ell$ with $\ell \approx \frac{c' \log k}{k^i}$ a prime and then retain those which cover $\mathbb{Z}_\ell$ under i-fold sums of elements in the sets and simultaneously the sum of the elements in the set avoid a fixed value. (The upper bound comes from a simple count on the number of edges.)

We also were able to find graphs $\mathcal{H}_i$ so that for fixed $i \ge 1$ and k growing we have $\pi(\mathcal{H}_i) = 1-\Theta\big(\frac{\log k}{k^i}\big)$. These graphs are closely related to the F_j so that essentially the same construction we found above worked for these graphs as well.

While we have made some progress and are within a log factor in the cases of which we are interested in, there is still some distance to go to understand the Turán density of F_i.

Clique Suspensions

For $r > s \ge 2$, let R be a set of $r-s$ vertices and H be a s-uniform hypergraph. A *suspension of* H, denoted S^rH, is an r-uniform hypergraph with the vertex set $V(H)\cup R$ and the edge set $\{F \cup R \colon F \in E(H)\}$. For a fixed H, the Turán density $\pi(S^rH)$ is a decreasing function of r. The case that H is a complete graph K_t is of particular interest. Bollobás and Leader and, independently, Bukh had previously asked whether $\lim_{r\to\infty} \pi(S^rK_4)=0$.

The following results on $\pi(S^rK_r)$ were obtained during the workshop. For $k=3$, we proved that $\frac{2}{er^2} \le \pi(S^rK_3) \le \frac{1}{r}$ for all large r. For $t \ge 4$, we proved that $\pi(S^rK_4) \ge \frac{c\sqrt{t}}{r}$ for some positive constant c. This lower bound is obtained via the following construction.

Construction: Let $V = V_0 \cup \cdots \cup V_{l-1}$, $V_i = V_{i,1} \cup \cdots \cup V_{i,s}$, where $l = \frac{r^2}{2}$ and $s < t-1$. Let $A \subseteq \mathbb{Z}_l$, $|A| \ge c\sqrt{l}$ be a Sidon set. Let H be an r-graph on vertex set V and edge set E, where each hyperedge has at most one vertex from each of $V_{i,j}$s and $f(E) := \sum_{i\colon\, V_i\cap E\neq\emptyset} i \pmod{l} \in A$.

John Goldwasser suggested a construction using Fano plane that shows that $\pi(S^3K_4) \ge \frac{1}{2}$. If certain designs exist, then his idea works for some other suspensions S^rK_t.

Tight Cycles

The *tight cycle* C_k^3 is the 3-graph with vertex set $\{v_0,\ldots,v_{k-1}\}$ and edges $\{v_i, v_{i+1}, v_{i+2}\}$ where i is a residue modulo k. In other words, we take some cyclic order on its k vertices while its edges are consecutive triples of vertices. Conlon (2010) showed that good bounds on the Turán function of tight cycles might improve best known bounds on the maximum number of edges in of the hypercube Q_n without the (usual) cycle C_{2l}, the latter question going back to Erdős to 1983.

The group proved that any 3-graph with n vertices and $\binom{n}{2}+1$ edges contains a tight cycle minus an edge. We still lack arguments that guarantee the last edge. This seems to be the stumblimg block for this and some related questions.

Independent Sets in Sparse Hypergraphs

The group came up with a short proof that every n-vertex linear r-graph has an independent set of size at least $cn^{1-1/(r-1)}(\log n)^{1/(r-1)}$ where $c > 0$ and $r > 2$. This was previously proved by the semirandom method by Ajtai, Komlós, Pintz, Spencer, and Szemerédi (1982). The general conjecture, which is the motivation for this approach, is to show that if F is a fixed r-graph and H is an n-vertex F-free r-graph of maximum degree d, then H has an independent set of size at least $cn(\log d)^{1/(r-1)}/d^{1/(r-1)}$. It remains open.

Concluding Remarks

The workshop was quite productive and stimulating. Some partial results have been obtained and a list of open problems, many of which are new, was compiled. We expect that a number of collaborative projects will result from this workshop.