Higher categories and topological order

organized by
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Workshop Summary

This workshop followed on from the online AIM workshop Fusion categories and tensor networks held in 2021, with the aim of exploring and solidifying the connections between the mathematical community studying higher categories, and the physicists exploring higher topological orders.

Each morning, there were two talks, with an additional, closing talk on Friday. The two opening talks were aimed at establishing a dictionary to connect the language used by the two sub-communities. This was extended on Wednesday morning, with an attempt to highlight and clarify some ambiguous language. Perhaps one of the most fruitful aspects of the workshop was a cross-disciplinary discussion on quantum cellular automata, which was also introduced on Wednesday.

The speakers were

- Mon:Ramona Wolf and Noah Snyder
- Tues-David Reutter and Maissam Barkeshli
- Wed-Colleen Delaney and Jeongwan Haah
- Thurs-Dmitri Nikshych and Theo Johnson-Freyd
- Fri-David Aasen, Ryan Thorngren and Zhenghan Wang

Problem list and initial progress

On Monday afternoon, there was a problem session moderated by Eric Rowell, which generated a list of around 50 open problems. The list was extended and amended throughout the week. Over the remaining afternoons, participants worked on the following 14 problems.

G-crossed braided categories with time reversal symmetry

This group investigated possible generalizations of the definition of G-crossed braided categories to include the physical structure of time-reversal symmetry, an anti-unitary operator. The physical challenge in understanding these structures is that it is unclear what it means to have a time-reversal symmetry defect in the bulk. Mathematically, this translates to determining if an anti-unitary structure can be put on the algebraic data of a G-crossed braided category. Maissam Barkeshli suggested that imposing this structure on G-crossed braided categories, as they are currently defined, might not be possible, implying that new mathematical structure is needed. One suggestion was to follow the construction of Jones, Penneys, and Reutter (arXiv:2009.00405) and define this new object as a functor from G into a 1-boring 3-category. However, instead of representing G as a 1-category with one object, we could represent it as a 1-category with two objects which get mapped to one another with anti-unitary elements of G and get mapped to themselves by unitary elements of G.

Order 2 elements of the Witt group and time reversal symmetry
This group checked a number of examples and found a theorem that stated the result in the special case of property S categories. The theorem should be that a category has time-reversal symmetry if and only if it has order 2 or 1 in the Witt group. One direction is clear: time-reversal symmetry implies order 2 or 1. We also discussed whether a result in this direction could have an application to constructing the conjectured time-reversal symmetry of F4 level 4 by first showing it had order 2 in the Witt group.

Quantify the failure of equivalent centers iff Morita equivalence for fusion 2-categories
Over an algebraically closed field, if two fusion 2-categories have equivalent centers, then they are Morita equivalent up to a class in the Witt group of nondegenerate braided fusion 1-categories. See "Morita equivalence and fiber functors for fusion 2-categories" below for related project.

Exact and separable algebras in non-semisimple tensor categories
This group first reviewed the proof that separable implies exact. Next, they found an example (in char. p) of an exact algebra that is not separable. They also obtained an intrinsic characterization of exact algebras. However, this involved a lot of data and this needs to be further simplified. Next step would be to look at the example where C=Rep(H) for H a Hopf algebra. In this case, an explicit characterization of exact algebras is known.

Floquet codes, condensations, and invertible domain walls
This group investigated which invertible domain walls arise from alternating condensations. The group quickly noted that every domain wall arises as a composite of condensations as in work of Kong, but interesting questions still remained about how to achieve Floquet codes from a sequence of algebras in a single anyon theory. A recurring theme of the unanswered questions was about the extent to which we could understand key error correction properties of floquet codes using only TQFT as opposed to a detailed lattice implementation. It was also noted that within the TQFT formalism classifying sequences of domain walls occurring at instants of time is equivalent to classifying so-called 1-foliated topological defect networks, where layers of topological defects are laid out along a spatial direction rather than time.

Skeletonization for fusion 2-categories
For many physical applications of (higher) fusion categories, it is convenient to present the category in terms of skeletal data. This group discussed the desiderata for this data in the context of fusion 2-categories. In particular, whether a basis of simple objects or Schur-connectivity classes should be used. They also discussed the '10j-symbols' with respect to these bases, and in what sense they can be invertible.

Morita equivalence and fiber functors for fusion 2-categories
This group investigated this problem with "2" replaced by various lower values of n. To make it nontrivial, they worked over various base fields. They found that the answer depended sensitively on the precise amount of (co)connectivity we assumed for our (lower) categories.

SPT for fusion category symmetries
This group investigated a possibility of introducing SPT symmetries that are not necessarily groups or supergroups. An interesting discussion between mathematicians and physicists

ensued about a motivation to study such phases of matter and how one could rigorously introduce a mathematical framework for these.

QCA/QC Witt

This group investigated the problem of whether the space of quantum cellular automata (QCAs) modulo finite depth quantum circuits (QCs) is equivalent to the Witt group of UMTCs. The group spent the first day learning more about Haah's "invertible algebras" and there was a detailed discussion about invariants for invertible algebras. The group made connections to AF C*-algebras and K-theory. The second day, the group tried a strategy to create a map from UMTCs to QCAs by reducing to a constrained topological Hilbert space of the Levin-Wen model for a UMTC, and then noting the algebra of string operators splits as a tensor product similar to Haah's invertible subalgebras. The "swindle argument" then produces a generalized QCA. The problem with this approach is that it was not seen how to promote this generalized QCA to an honest QCA while preserving the invertible subalgebra. On the third day the group discussed extending the concept of QCA to subalgebras of the full algebra of operators on a tensor product of qudits, such as those that commute with a global symmetry. For such constrained QCAs it is expected that the classification becomes richer even in one dimension, including dualities such as Kramers-Wannier. The group discussed how to adapt the boundary algebra approach to this setting.

Pivotal ENO extension theory

This group defined a pivotal version of the Brauer-Picard groups (PivBrPic(C)). To do so, we used relative Serre functors to define the notion of a pivotal bimodule category. Using the pivotal Brauer-Picard groups, we were able to conjecture a characterization of pivotal structures on a G-extension C (with 2-functor F from from BG to BBrPic(C)) in terms of lifts of F to a 2-functor from BG to BPivBrPic(C). Further, we have a conjectural description of obstructions to such lifts in terms of twisted cohomology of G.

Modules over Rep SU(2) and their duals

In the case where the module category comes from a discrete group, we don't need to think about some annoying "size" issues which would require analytic or algebraic geometry techniques. In this case everything turns into some combinatorics related to double-coset spaces H(2)/H. This group looked at the example H(3)/H where H is the icosahedral rotational group in a lot of detail. This is all very classical mathematics, but it's a bit involved to really understand it correctly.

Representations of motion groups

This group discussed the general problem of constructing motion group representations from (3+1)TQFTs, concluded that this was hard, in practice. On the other hand certain motion groups with known presentations (cabled torus knots, for example) have yet to be explored. This group had some ideas–mostly of the form: extend braid group representations.

W* condensations

This group investigated how to produce the symmetric monoidal 2-category of von Neumann algebras from the symmetric monoidal 1-category of Hilbert spaces by some sort of formal completion. The group investigated the projective tensor product of Banach spaces and whether the predual of a von Neumann algebra has a non-unital condensation algebra structure. The answer was positive for B(H) and $^{\infty}$, but the answer remained inconclusive for $L^{\infty}[0,1]$.

Integral modular categories with fixed prime divisors

The problem of classifying modular categories with integer FP-dimensions has a number theoretical approach when the dimension of the category has a fixed finite set of prime divisors. This leads to a certain Diophantine equation where the collection of solutions is severely reduced by the restricted set of values. In particular one might improve on the usual doubly-exponential bound given by Sylvester's sequence by means of a modified greedy algorithm.