Higher-dimensional contact topology

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Workshop Summary

This document summarizes the activity and focus of each of the working groups during the AIM Workshop " Higher-dimensional contact topology" (April 15 to April 19, 2024).

Reports on working groups

Group 1 "Taxonomy of contact submanifolds via mushrooms and plugs"..
The group focused on two questions. First, understanding the details of the techniques of Honda-Huang in their development of convex hypersurface theory in higher dimensions. In particular, this includes a precise comparison to the dynamics of σ-plugs, as developed by Eliashberg-Pancholi. Second, the group focused on understanding whether convex hypersurface theory could be you to establish an h-principle for contact submanifolds. Specifically, for codimension-2 contact submanifolds and whether the Honda-Huang higher-dimensional mushroom can be used to resolve the (local) extension problem given by the standard wrinkle.

Group 2 "Explicit contact handlebodies"..
The group "Explicit contact handlebodies" was interested in studying the family of contact manifolds $(S^5, \xi_n) = \mathrm{OB}(T^*S^2, \tau_S^{2n+1})$, where τ is a positive Dehn twist around the 0-section. It is well-known that these are exotic tight contact structures on S^5. By removing two small Darboux balls from one of these contact spheres, one obtains a contact structure on $S^4 \times [0,1]$ with standard convex boundary. The specific goal was to compute bypass decompositions of these layers. Though the group did not completely achieve this goal, the participants working in it identified stepping-stone problems that should give insight, or at least inspiration, for the main problem. For instance, in dimension 3, consider the overtwisted S^3 given by $OBD(T^*S^1, \tau^{-1})$. One may ask the same question as above (identify a bypass decomposition between two Darboux balls). The group successfully did the "bypass calculus" to show that (two overtwisted bypasses) = (neg stab. + pos stab.). Next, observe that this overtwisted sphere is given by contact (+1) surgery on the Hopf link. The group would next like to see the two overtwisted bypasses between Darboux balls indicated above directly in the surgery diagram. Even in dimension 3, it was not immediately clear how to see the bypasses. To see why this problem may be useful for identifying bypasses in the higher-dimensional examples, notice that the above contact spheres are obtained from contact (-1) surgery along a spun 2n-component Hopf link.

Group 3 "Symplectic fillings of contact submanifolds"..
The general question is to understand the symplectic fillings of a codimension 2 contact submanifold. One sample question, due to Roger Casals, is if the symplectic fillings of the

standard contact 3-sphere in the standard contact 5-sphere inside the standard symplectic 6 ball are unique symplectically.

On the topological side, the answer is affirmative in the following sense. Any symplectic filling of the standard $2n-1$ contact sphere in the standard $2n+1$ contact sphere inside the standard $2n+2$ ball is an unknotted $2n$ ball. This is obtained by holomorphic planes asymptotic to the shortest Reeb orbit and the Siefring intersection theory where we view the symplectic filling as a holomorphic divisor in the $2n+2$ ball. A similar argument can be applied to contact manifolds pairs $(\partial(V\times\mathbb{D}\times\mathbb{D}), \partial(V\times\mathbb{D}))$. However, it seems to be beyond the current technology to verify if the filling is symplectically unique. However, it should be possible to establish the invariance of some symplectic invariants (e.g. holomorphic curve invariants) of the submanifold filling.

The group then turned to the other side, namely examples with different symplectic fillings. The group is planning to work on discussing the following cases by imposing stronger restrictions:

(1) Contact submanifolds with fillings that are different as symplectic manifolds. Let $f : V \to \mathbb{C}$ be a Lefschetz fibration with regular fiber Σ and monodromy ϕ, then we consider the trivial fibration $f + z : V \times \mathbb{C} \to \mathbb{C}$. Using this, the inclusion $\Sigma \subset V$ allows an extension of ϕ to become trivial in the symplectic mapping class group V. Using this we can embed a symplectic fibration over a once punctured torus with fiber Σ and monodromy around the merdian is ϕ to the trivial fibration over the punctured torus with fiber V.

  (a) By choosing suitable ϕ, we can embed fillings with different topology into the trivial fibration with the same contact boundary.

  (b) Similarly, it is likely that we can embed fillings of the same topology but different symplectic structures into the trivial fibration with the same contact boundary. Potential examples are those using ϕ that are smoothly trivial, while the symplectic fibration has non-trivial Fukaya category by Kartal's work. But it seems we can only differentiate them as graded symplectic manifolds.

(2) Contact submanifolds with fillings that are same as symplectic manifolds, but are smoothly knotted. For this, we only have low dimensional examples and knottedness is detected by the simplest invariant, the relative homology class.

  (a) Let L be a Lagrangian disk that intersects a Lagrangian sphere S, we can push off L to be a symplectic submanifold L', then L' and $\tau_S(L)$ often have different relative homology classes, where τ_S is the Dehn-Seidel twist around S.

  (b) This might be a special case of above, we consider T^*S^2, the shortest Reeb orbit in the perturbed Boothy-Wang contact form will bound two holomorphic disks, whose relative homology class is differed by the zero section.

(3) Contact submanifolds with fillings that are same as symplectic manifolds, have the same isotopy class as smooth embeddings, but are symplectically knotted. This is the most interesting version of the question, but the group made no progress on this.

One of the challenges for those question is to construct multiple symplectic fillings of a contact submanifold, let along making them knotted.

The following people participated the discussion of the questions: Sheel Ganatra, Fabio Gironella, Eric Kilgore, Oleg Lazarev, Agniva Roy, Lisa Traynor, Igor Uljarević, Zhengyi Zhou.

Group 4 "Tightness criterion via CHT".
In 3-dimension there is a very simple tightness criteria for neighbourhoods of surfaces given by Giroux: the contact structure is overtwisted in the neighbourhood of a convex surface if and only if there is a component of the dividing curve that bounds a disc on one side and something else on its other side. The goal of our group was to generalise this criteria for higher dimensions.

This criteria feels a bit too special for 3-dimension, so instead the group searched for other 3-dimensional tightness criteria that are more generalisable. The one the group found the most promising is the following: the contact structure is overtwisted in the neighbourhood of a convex surface if and only if it is obtained from some convex surface via an overtwisted bypass attachment.

Luckily Honda-Huang described a bypass in higher dimension that produces a convex hypersurfece with overtwisted neighbourhood. The group named this bypass: the obviously overtwisted bypass (OOB). Eventually, after understanding the proof for this statement this group managed to generalise this criteria using flexibility techniques: the contact structure is overtwisted in the neighbourhood of a convex hypersurface if and only if it is obtained from some convex hypersurface via an obviously overtwisted bypass attachment. The group spent the rest of the meeting trying to convert this to a checkable criteria in dimension 5 using the reformulation of Avdek.

Group 5 "Conformal symplectic geometry".
The group had a few initial aims:

(1) To make progress towards understanding which contactomorphisms produce convex mapping tori.

(2) To see whether there is an interesting relationship between the locally conformal symplectic (lcs) structure one can place on the mapping torus associated to a contactomorphism and the convexity or non-convexity of that mapping torus.

(3) To list and compare all possible notions of overtwistedness for lcs structures and determine whether each one abides to an h-principle.

(4) To discuss non-squeezing, orderability and capacities for lcs manifolds.

This group ended up focusing mostly on the first question, for which interesting phenomena occur, already in dimension 2, that is for contactomorphisms of the circle. The aim became to try to find conditions on a contactomorphism that both prevent convexity and are stable under smooth approximations. One idea was to find an example of a contact Anosov flow. The groups will pursue its work after the workshop with planned discussion on the lcs questions as well.