Graph Theory: structural properties, labelings, and connections to applications

organized by
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Workshop Summary

Introduction

Our graph theory workshop was held July 22-26, 2024 at AIM on CalTech's campus. We had 32 participants from across North America, Europe, and Asia. Our group included many mid-career faculty looking to restart their research program, some post-docs and junior faculty, and some senior faculty. Moreover, this AIM workshop supported the AWM Women in Graph Theory and Applications community and 75% of the participants were women.

On the first day of the workshop, the group proposed possible research problems in three main areas: graph labeling/graph coloring, variations of zero forcing/power domination, and structural properties and connections. Over the course of the week, 9 groups were formed to look deeper into some of these questions. Most groups are continuing their collaboration after the workshop. There is a plan to have a special issue of *The American Mathematical Monthly* devoted to the work produced from our workshop. We also plan to run a special session at a future conference to highlight results stemming from this workshop.

The week can be summarized by this word cloud generated on the last day of the workshop:

We want to thank AIM for providing the space and resources for us to gather and collaborate.

Group Reports

(1) **Edge Total Magic Labeling**

An edge magic total (EMT) labeling of a graph $G = (V, E)$ is a bijection from the set of vertices and edges to the set of integers defined by $\lambda : V \cup E \to \{1, 2, \ldots, |V| + |E|\}$ so that for every edge $xy \in E$, its weight defined by $w(xy) = \lambda(x) + \lambda(y) + \lambda(xy) = k$, for some integer k. An EMT labeling is said to be a super EMT labeling if the vertices are labeled with the smallest possible integers $1, 2, \ldots, |V|$. During the week at AIM workshop, we studied the open problem of EMT labeling on the disjoint union of cycles. Note that for 2-regular graphs, a graph has an edge magic total labeling if and only if it has a vertex magic labeling. Therefore, we studied the paper "Vertex magic total labelings of 2-regular graphs" by Cichacz, Froncek, and Singgih, where they conjectured that for even values of n, $2C_n$ has a vertex magic total labeling with $k = 5n + 2$ where k is the magic constant. We understood their constructions and how they generalized their labeling to a larger graph. We also studied the paper "A magical approach to some labeling conjectures" by Figuero-Centeno et. al. We narrowed down the open problems to $2C_n$ where $n = 2p$ ($p > 7$

prime), $n = 2^q, q \geq 4$ and n odd, $n \equiv 3, 9 \pmod{12}$. In addition, we made some useful observations.

(2) **Proportional Zero Forcing**

Our group introduced a new variant of the zero forcing coloring game on graphs. Let $G = (V, E)$ be a graph and assume that each vertex is colored either blue or white, but not both. In *standard zero forcing*, we start with an initial subset of blue vertices and obey the following color change rule: if at time t, v is blue and v has exactly one white neighbor x, then the color of x changes to blue. We say $F \subseteq V$ is a *zero forcing set* of G if the vertices in F are initially assigned the color blue, and after repeated application of the color change rule, all vertices in G become blue. The *zero forcing number* $\mathrm{Z}(G)$ of G is the minimum cardinality of a zero forcing set.

For example, $\mathrm{Z}(K_n) = n - 1$ for the complete graph K_n since any vertex has at least 2 white neighbors unless all but one vertices are initially colored blue.

In *proportional zero forcing*, we assume that the ability of a blue vertex to turn another vertex blue is no longer binary but instead determined by a quantity that degrades through the transmission process. Let $G = (V, E)$ be a graph and assume that each vertex is colored either blue or white, but not both, and each vertex y has an assigned weight, given by $w(y)$. The weight function and coloring will change according to the *proportional color change rule*: at time t, for a white vertex x with any blue neighbor v such that x is the only white neighbor of v, each such v adds $\alpha \cdot w(v)$ to $w(x)$, and if the total new weight of x is at least β, then the color of x changes to blue. We assume that the *effective transmission proportion* $\alpha \in [0, 1]$ and the *effective transmission threshold* $\beta \in [0, 1]$ are constant for all transmissions and fixed in time. The *proportional zero forcing number* of G for α, β, denoted by $\mathrm{Z}_{\alpha,\beta}(G)$, is the minimum size of an initial blue set F such that repeated application of the proportional color change rule results in all vertices of G being blue.

Returning to our earlier example, we note that $\mathrm{Z}_{\alpha,\beta}(K_n) = n - 1$ if $(n-1)\alpha \geq \beta$ and $\mathrm{Z}_{\alpha,\beta}(K_n) =$

(3) $$g(e) = \begin{cases} f(e) & \text{if } f(e) \neq j \\ i & \text{if } f(e) = j \end{cases} \text{ for all } e \in E(G).$$

If this implies there exists $u \neq v$ such that $S(u) \neq S(v)$ for all $i \neq j$, then f is a minimal VDEC.

For graph G, denote the minimum k such that G admits a VDEC with k colors as $\chi'_S(G)$. This chromatic VDEC number has been studied before. We defined the achromatic VDEC number to be the maximum k such that G admits a minimal VDEC with k colors as $\Psi'_S(G)$. We worked on determining $\Psi'_S(G)$ for several classes of graphs. Furthermore, we realized we could modify the definition to create related VDEC measures. For example, rather than using the edge coloring to create unique sets of the incident edges for each vertex, we could define $S(v)$ to be the multi-set of colors incident to vertex v where we still want each $S(v)$ to be unique. The minimum k such that G admits a multi-set VDEC with k colors is denoted as $\chi'_{MS}(G)$ and the maximum k such that G admits a minimal multi-set VDEC with k colors is $\Psi'_{MS}(G)$. We then began to consider ways we could leverage other known graphical results, such as the existence of a 1-factor or not, to help determine $\Psi'_{MS}(G)$.

Non-trivial square graphs

Motivated by recent work of Garg, Raymond, and Redlich, this group explored which graphs and families of graphs admit a non-trivial square root. This line of research has ties to Sidorenko's conjecture, in particular, whether or not the Cauchy-Schwarz sum-of-squares approach is a feasible proof technique for some specific bipartite graph families. Past work has classified a number of bipartite graph families for which Sidorenko's conjecture is known. During the workshop, group participants familiarized themselves with the problem, its motivation, and known results. The group then explored additional graph families and proved necessary and sufficient conditions for some families of graphs to admit non-trivial squares.

Graph homomorphisms

Graph homomorphisms—functions from between the vertex sets of graphs that preserve adjacency — provide one lens through which graph labelling can be viewed. We consider an extremal enumerative question in the field of graph homomorphisms: for each graph H and integer n, which tree on n vertices admits the fewest homomorphisms to H? In other words, which tree admits the fewest labelings, where the vertices of H are the labels, and adjacent vertices in the tree must be given labels that are adjacent in H? When "fewest" above is replaced by "least", the answer is well known, but there is a lot still to be discovered in the minimization version of the problem.

We studied a specific construction in the literature, and sketched how to generalize it from one specific instance of the problem (where it shows that for $n = 7$ the problem will not admit a nice uniform answer) to an infinite family of instances. We obtained a new infinite class of graphs for which the minimizing tree (for every n) is the path. We discussed a number of concrete questions and conjectures around the problem, perhaps the most appealing of which is the conjecture that for every H, the path is the minimizing tree for all sufficiently large n.

The group has started meeting regularly via zoom to continue this work.

Neighborhood Balanced Colorings:

A neighborhood balanced coloring, introduced by Freyberg and Marr in the paper "Neigborhood Balanced Colorings of Graphs" in *Graphs and Combinatorics*, is a vertex coloring of a simple graph G by two colors such that every vertex in G has an equal number of neighbors of each color. During the week several variations and generalizations of neighborhood balanced colorings were proposed and studied. One variation included extending neighborhood balanced colorings to more than two colors, particularly looking at 3-colorings on circulant graphs. Another variation of the problem that became the focus of much of the work was closed neighborhood balanced colorings. A *closed neighborhood balanced coloring* (CNBC) of a graph is a red-blue coloring of its vertices such that the closed neighborhood of every vertex contains an equal number of red and blue vertices. If a graph G admits such a coloring, we say G is a CNBC graph. The work on CNBC graphs involved establishing the necessary notation, definitions, and preliminary results, as well as proving results for specific families of graphs, trees, and graph operations.

Radio Graceful Graphs:

For a simple connected graph G, a labeling $f : V(G) \to \mathbb{Z}_+$ is a radio labeling of G if it satisfies $|f(u) - f(v)| \geq \mathrm{diam}(G) + 1 - d(u, v)$ for all distinct vertices u, v in $V(G)$. A radio labeling f of a graph G is a consecutive radio labeling if $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$. A graph for which a consecutive radio labeling exists is a radio graceful graph.

In this workshop, we worked to determine graphs which are radio graceful. In particular, this group focused on looking at Cartesian products of some graphs. In the process of looking at these Cartesian products, we introduced the concept of circular radio graceful graphs. In addition, we worked to develop constructions of radio graceful graphs that are diameter 2. The group continues to meet to work on each of these projects connected to radio graceful graphs.

Induced Subgraphs

Let $d < k < n^2$ and let G be a bipartite graph with partite sets X and Y where the vertices in X correspond to all possible d subsets of $[n]$ and the vertices in Y correspond to all possible k subsets of $[n]$. For $x \in X$ and $y \in Y$, the edge $xy \in E(G)$ if $l(x) \subseteq l(y)$ where $l(x)$ and $l(y)$ are the subsets of $[n]$ that correspond to x and y, respectively. Our group met to discuss the possible induced subgraphs of this graph G. Our group met just one day during the workshop and explored the answer to this question for small values of k, d, and n.

Distance Antimagic

A graph G is called *distance magic* if there exists a bijection $f : V(G) \to \{1, 2, \ldots, v\}$ such that at any vertex x, *the weight* of x, $\omega(x) = \sum_{y \in N(x)} f(y)$ is constant, where $N(x)$ is the open neighbourhood of x, i.e., the set of vertices adjacent to x. A graph G is said to be *distance antimagic* if there is a bijection $f : V(G) \to \{1, 2, \ldots, v\}$ such that for every pair of distinct vertices x and y applies $w(x) \neq w(y)$. In this case, the bijection f is called *antimagic labeling* of G.

Our group met to investigate when the Cartesian product of two graphs is distance antimagic for different classes of component graphs. The component graphs considered included complete graphs, complete bipartite graphs, wheel graphs, cycle graphs, and path graphs. We also examined the lexicographic product of different classes of graphs. Several of our results start with at least one graph that is distance magic, and exploit that labeling to obtain an antimagic labeling of the corresponding product graph.