Graph and hypergraph limits

organized by
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Workshop Summary

1 Overview

The purpose of this workshop, sponsored by AIM and NSF, was to bring together a diverse group of researchers that are interested in graph limits with the purpose of sharing expertise, learning about recent developments, and initiating new collaborative projects. The mathematical interests of the attendees were remarkably varied: probability theory and statistics, property testing in computer science, applications of flag algebras to extremal combinatorics, logic and universal countable graphs, graphs homomorphisms, exponential random graphs, and others.

We had ten lectures on different aspects of graph limits. A general introduction was given by László Lovász. Persi Diaconis and David Aldous presented the theory of exchangeable distributions and how it is related to graphons. Omer Angel talked about limits of bounded degree graphs. Cameron Freer presented extensions of the Petrov-Vershik construction of invariant measures on the Rado and Henson graphs via graph limits. Sergey Norin described how flag algebras led to solutions of a few long-standing open questions of extremal combinatorics. Balázs Szegedy discussed graphons as kernel operators and the spectral regularity lemma. Jaroslav Nešetřil looked at possible asymptotic logarithmic subgraph densities in a class of graphs. Po-Shen Loh talked about analytic methods in graph theory. Sourav Chatterjee showed how graphons appear in the study of random exponential graphs.

Also, there was an open problem session on Monday afternoon. The organizers complied a list of some of the problems presented there. This list should be available from the AIM's website.

Afternoons were dedicated to discussions, collaboration, and group work. Here are some of the projects that were considered during the workshop.

2 Selected Projects

Terms and concepts that we are not defined here can be found in an excellent survey by Lovász [lovasz:09].

2.1 Characterization of Positive Graphs

A graph H is *positive* if for every symmetric measurable function $W : [0,1]^2 \to [-1,1]$ we have $\hom(H,W) \geq 0$, where

$$\hom(H, W) = \int_{[0,1]^{V(H)}} \prod_{ab \in E(H)} W(p(a), p(b)) dp.$$

Problem 1 (Lovász [lovasz:08:u]) *Characterize positive graphs.*

It is possible that a graph H is positive if and only if its vertices can be partitioned into three groups (I, A, B) so that I is an independent set, there is no edge between A and B, and there exists an isomorphism between the subgraphs spanned by $I \cup A$ and $I \cup B$ which fixes I.

The above condition is clearly sufficient because in this case, given the image of I, the average homomorphism is a square of some expression. An earlier (unpublished) result of Lovász states that if H is positive, then there exists a homomorphism of H into itself so that the preimage of each edge has even even size.

It is not hard to show that H is positive if and only if $t(H, W) \geq 0$ for every ± 1-valued stepfuction W. We proved that H is positive if and only if $\hom(H, W, I_0, \ldots, I_k)$ is positive for every measurable partition $[0, 1] = I_1 \cup \cdots \cup I_k$ with $k \geq \Delta(H)$. Here $\hom(H, W, I_0, \ldots, I_k)$ is the same as $\hom(H, W)$ except a vertex of degree m in H has to be mapped to I_m. The latter result seems to be quite useful. It has the following two corollaries.

Corollary 2 *If H is positive, then for each k, the subgraph of H spanned by all nodes with degree k must also be positive.*

Corollary 3 *For each odd k, the number of nodes of H with degree k must be even.*

Using these tools and a very simple computer program for finding counterexamples, we checked the conjecture for all graphs on at most 6 vertices. Whenever we looked for a nontrivial counterexample for a graph, we have found it as a 2-step function except in the case of $K_{2,2,2}$, where we proved that it is positive on all 2-step functions, but not on some 3-step function. Interestingly, the result that $K_{2,2,2}$ is not positive answered Problem 23 in [lovasz:08:u].

We proved that some other graphs are not positive: the cube and every complete partite graph with at least 4 parts. Also, we showed that a graph with exactly two connectivity components is positive if and only if the two components are isomorphic or both are positive.

2.2 Simpler Construction of Henson Graphon

Recall that the *Henson graph* is the countable ultrahomogeneous universal triangle-free graph, which can be constructed as the limit of all finite triangle-free graphs. Define a *Henson graphon* to be a symmetric measurable function $W : [0, 1]^2 \to \{0, 1\}$ such that the countable W-random graph is almost surely isomorphic to the Henson graph.

Petrov and Vershik [petrov+vershik:10] provided a construction of a Henson graphon, and in fact described all such graphons up to measure-preserving isomorphism, and also showed that one need not consider greyscale graphons, i.e., maps into $[0, 1]$ rather than $\{0, 1\}$.

We examined the generalization of this construction by Ackerman, Freer, and Patel to arbitrary countable relational structures having trivial group-theoretic definable closure. We noted that their method can be seen as interleaving the construction of a limit step-function along with steps that subdivide each region ever more finely (to witness all extension axioms, e.g., to preserve universality in the case of the Henson graph). This construction creates an expanding "checkerboard" where the constant value of each rectangle is determined once and for all.

We also provided a new construction, in the case where the countable structure is ultrahomogeneous, and hence a limit. Rather than tiling an increasingly large region of the square, we describe a sequence of step-functions defined on the entire unit square, but such that we keep changing our mind on ever smaller regions as the construction proceeds, so that the sequence is seen to converge to a graphon that induces the desired structure. The sequence of step-functions is defined in accordance with an explicit construction, and makes use of the fact that definable closure is trivial (i.e., the age has strong amalgamation).

We also explored when it is possible to have any greyscale regions (as cannot happen for the Henson graph); in general this should not be possible for graphs that are universal among those omitting a class of forbidden subgraphs, but there may be others.

2.3 Countable W-Random Graphs

Omer Angel asked at the workshop how many isomorphism classes of countable graphs can be in the support of a W-random graph. It was noted that when the graphon is given by a countable partition of $[0, 1]$ such that each rectangular region in the product of the partition with itself is constant-valued, there must be one isomorphism class.

By a result of Lovász and Szegedy [lovasz+szegedy:PosSatz] there must be either one or uncountably many isomorphism classes, because the W-random graphs are ergodic. If a countable number (more than 1) of isomorphism classes were in the support, at least two isomorphism classes would occur with positive probability, and one could write the distribution of the W-random graph as a nontrivial mixture.

Gábor Kun gave a construction of a graphon W such that the support of the W-random graphon has continuum-many isomorphism classes of countable graphs. The graphon W is the characteristic function of the countable union of disjoint, axis parallel rectangles. It has the following properties (these imply that there are continuum isomorphism classes). For every $x, y \in [0; 1/2], x \neq y$ there exists a finite graph H such that one of them is contained by a subgraph isomorphic to H, but the other is not contained by any subgraph isomorphic to H. Moreover, every dense induced subgraph of W containing that vertex contains such an isomorphic copy of H. And for almost every $x \in [0; 1/2]$ there is a finite graph H such that for every dense subgraph of W containing x there is a subgraph isomorphic to H and containing x, and if $y \in [1/2; 1]$ is contained by a subgraph of W isomorphic to H then there is no isomorphism between the two copies of H mapping x to y.

2.4 Representing Equivalence Classes of Graphons

We would like to select a representative from each equivalence class of graphons in a measurable way (without using the axiom of choice). We tried to use the spectral decomposition of a graphon as a kernel operator, see [szegedy:11].

The spectra is discrete and the multiplicity of any non-zero eigenvalue is finite. If there are no multiple non-zero eigenvalues then the representative can be defined by ordering the eigenvalues and choosing the eigenfunctions to be increasing in the first eigenvalue and so that the first coordinate in the next ones is as big as possible (avoiding the sign uncertainty). If there are multiplicities, we need to use an auxiliary Poisson expansion from the eigenfunctions to provide a rearrangement (work still in progress).

Bibliography

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