Surfaces of infinite type

organized by

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Workshop Summary

The goal of this AIM workshop was to bring together the researchers that work on the emerging field of surfaces of infinite type, and in particular those working on big mapping class groups and infinite translation surfaces. Our aim was to form a community of mathematicians by bridging the gaps between the groups of individuals working on various aspects of these topics, and to put forth a list of some of the most interesting and important open questions regarding infinite-type surfaces. These goals were certainly fulfilled through this workshop.

1. Schedule of the week

As usual for AIM workshops, the program of every morning consisted of two talks, ranging from introductory talks to expositions of more recent developments that provided some of the motivation for problems that were posed for the working groups. The speakers and topics of the talks were the following:

- Javier Aramayona: Introduction to big mapping class groups
- Ferrán Valdez: Infinite translation surfaces in the wild
- Alden Walker: Actions of big mapping class groups
- Kathryn Mann: Large scale geometry
- Carolyn Abbott: Actions of big mapping class groups on relative arc complex
- Danny Calegari: The shift locus: Complex dynamics and big mapping class groups
- Nicholas Vlamis: Curious constructions of Katsuhiko Matsuzaki
- Gabriela Weitze-Schmithüsen: Veech groups of infinite translation surfaces
- W. Patrick Hooper: Thurston's construction in infinite genus
- short talks by Federica Fanoni, Spencer Dowdall, Marissa Loving / Justin Lanier, Alex Rasmussen, Ty Ghaswala, and Israel Morales

On Monday afternoon, we collected a large list of open questions building off of foundational results in this field, with a nice range in level of difficulty. For the rest of the afternoons during the week of the workshop, the participants worked in groups on these problems. The groups were very dynamic, with participants changing their groups quite a bit throughout the week. A few of the problems were resolved at the workshop and new problems were added during the week. This exemplifies the great exchange of mathematical ideas that occurred.

2. Main topics of discussions

Numerous problems were discussed during the workshop. In broad strokes, we can group the problems into five categories. We will describe these categories in more details below. A complete list of all the problems can be found here: http://aimpl.org/genusinfinity

2.1 Classification of elements of big mapping class groups. A fundamental theorem in the study of finite-type surfaces is the Nielsen-Thurston classification of mapping classes. The theorem states that every mapping class is either periodic, reducible (fixes a multicurve), or pseudo-Anosov. Pseudo-Anosov mapping classes exhibit a number of remarkable algebraic and dynamical properties, and in many senses, they are the typical elements. The most valuable aspect of the classification is that it gives a simple characterization of pseudo-Anosov mapping classes: namely, a mapping class is pseudo-Anosov if and only if it is irreducible and has infinite order.

An obvious question is whether big mapping class groups admit a similar classification theory. This problem appears quite open as it is not obvious what the correct answer should be, or even what the correct definitions should be. At the moment, it is crucial to just construct interesting examples of mapping classes of surfaces of infinite type. During the workshop, Pat Hooper explained a generalization of Thurston's construction of pseudo-Anosov mapping classes to surfaces of infinite type, and during a couple of the afternoons, a group explored various possible notions of irreducibility. Hooper's examples provided some test cases of what the correct definition should be. It turns out the natural definition of not fixing a multicurve is not enough to guarantee the type of dynamical properties that one would want from an irreducible mapping class. Though there wasn't enough time to make serious progress, a fruitful conversation was initiated during the workshop. Some members of the group have expressed intentions to continue the discussion after the workshop.

Another point of view of Nielsen-Thurston classification comes from the curve graph. From this point of view, a mapping class is pseudo-Anosov if and only if it is an loxodromic isometry of the curve graph. All other types act elliptically. In the infinite-type setting, the curve graph is unfortunately a finite diameter graph. Instead, we study the ray and loop graphs for these surfaces which are both hyperbolic and infinite diameter. During the workshop there was a large group of participants who worked on classifying loxodromic isometries of the ray/loop graph. The group first focused on understanding existing constructions of loxodromics, and then attempted to generalize these examples in several ways. For example, the group tried to come up with criteria under which the limit of finite-type pseudo-Anosov homeomorphisms converge to an infinite-type homeomorphism that acts loxodromically on the ray graph. There were many fruitful discussions about possible approaches to a full classification, and there are at least two subgroups that were formed from this group that are currently working on this problem. Later in the week this group also discussed the problem of whether or not there exist elements of the mapping class group that act parabolically on the ray/loop graph, which does not occur in the finite-type setting. Though there was less time allotted to discussing this problem, there was a consensus that it might be possible to construct such an element and there are plans to discuss this problem further once the more pressing questions regarding loxodromic elements are addressed.

2.2 Large scale geometry of big mapping class groups. One of the main questions of the workshop was to investigate whether big mapping class groups have a well-defined metric up to quasi-isometry. As big mapping class groups are not locally compact or compactly generated [1], standard geometric group theory doesn't apply. Fortunately, recent work of Rosendal establishes that many techniques of classical geometric group theory could be extended to a wider class of groups, namely those with a local "coarse boundedness" property. In work-in-progress, Kathryn Mann and Kasra Rafi have shown that many big mapping class groups have a well-defined quasi-isometry type. The focus of their work is on those groups with trivial QI type, i.e. those that are globally coarsely bounded. As a consequence, they recover some existing results on bounded orbits for actions on curve graphs. However, a complete QI classification is still far from completion and there are many interesting open problems. A group was formed on Tuesday and Wednesday afternoon to discuss these questions and to play with some examples and test cases. These discussions led to a set of notes written up by Nicholas Vlamis, which can be found here: http://qcpages.qc.cuny.edu/∼nvlamis/Papers/AIM_Notes.pdfhttp://qcpages.qc.cuny.edu/∼nvlamis/Papers/AIM_Notes.pdf

2.3 Teichmüller theory and other tools for surfaces of infinite type. In the study of surfaces of finite type, there are now many tools at our disposal, such as Teichmüller theory, train tracks, measured foliations and laminations, geodesic currents, earthquakes and so on. Great strides have been made toward generalizing these tools to infinite-type surfaces by Dragomir Saric and his collaborators, but many questions remain open. Developing these tools would have wide ranging applications, such as Nielsen-Thurston classification and Nielsen realization. During the workshop, Carolyn Abbott and Nicholas Vlamis gave talks explaining what is known about these tools and what are some of the difficulties in extending them to infinite-type surfaces. Though there wasn't a group dedicated to these topics per se, several groups naturally touched on one or more of these tools while working on their own problems. We expect progress to be made in the future as many of the groups have expressed they would continue their discussions.

2.4 Algebraic and topological properties of big mapping class groups. One of the motivating questions in the study of big mapping class groups is better understanding the algebraic and topological structures of these groups and how each of these affects the other. In the introductory talk on the first day of the workshop, Javier Aramayona touched on some of the existing foundational results regarding the algebraic and topological structures of big mapping class groups, for example the topological generating set given by Patel-Vlamis, work of Bavard-Dowdall-Rafi regarding Ivanov's meta-conjecture in the context of big mapping class groups, and interesting maps from big mapping class groups to $\mathbb{Z}$ constructed by Aramayona-Patel-Vlamis. There were two different groups of people that worked on addressing the question of whether or not big mapping class groups are Hopfian or co-Hopfian, and the second group believes that they have answered the Hopfian question. There were several other groups that worked on questions of this nature, studying homomorphisms of big mapping class groups. One example of a hard open questions of this type is whether big mapping class groups are automatically continuous. Due to the fact that big mapping class groups (and their closed subgroups) are Polish, homomorphisms to $\mathbb{Z}$ are automatically continuous. We would like to be able to replace $\mathbb{Z}$ with any separable group and make the same conclusion. Since big mapping class groups are studied as topological

[1] On day two of the workshop, one of the groups worked on the question of whether or not big mapping class groups are compactly generated, and concluded that they are not. Their proof has been recorded in Vlamis' notes.

groups, an automatic continuity result would prove invaluable to studying homomorphisms of these groups. Though there was not substantial progress made on this problem during the week of the workshop, there are many individuals who seem interested in continuing to work on this problem in the future and it was clearly labeled as one of the big open questions in this area.

2.5 Translation surfaces of infinite type and their Veech groups. Surfaces of infinite type can also be equipped with more rigid structures, in particular with translation structures. In the finite type case, translation surfaces arise naturally from Teichmüller theory, algebraic geometry, and dynamical systems, for example. The dynamical properties and the groups of symmetries (called Veech groups) of finite translation surfaces have been studied extensively and provide many tools to other fields. In this spirit, a generalization of Thurston's construction of pseudo-Anosov mapping classes to infinite translation surfaces was discussed as a tool for the questions in Section 2.1.

Two groups also studied the Veech groups of infinite translation surfaces. Whereas even for finite translation surfaces, it is not known which groups can arise as Veech groups, this question is settled for tame infinite translation surfaces with one end or uncountably many ends. One working group constructed two families of translation surfaces with two ends that have interesting Veech groups. Another working group started studying the Veech groups of all covers of the Chamanara surface, an infinite translation surface with a large Veech group. This group has continued its work and hopes to shed more light on Veech groups of covering constructions with this example.

3. Outcomes and outlook

Working on problems together for one week has brought together the participants of the workshop to form a community of researchers working on surfaces of infinite type. More than half of the participants were graduate students and postdoctoral fellows whose enthusiasm for working in this area is both driving the field forward at a rapid pace and creating an exciting and welcoming environment for those who are looking to break into this area of research. The diversity of the group, in particular regarding mathematical backgrounds and seniority, has helped to create a very fruitful atmosphere. Participants called the workshop "extremely productive", "energetic", and an "affirming experience".

Many of the working groups are continuing their projects, the first notes have already been written (see Section 2.2), and other groups are writing up their results at the time of writing this report. We expect also new collaborations to form on other questions that have been collected in our problem list but for which there was not enough time allotted. Two participants have also expressed interest to organize further conferences on the topic of infinite-type surfaces – one in Europe and one in Latin America. The authors of this report are certainly looking forward to these conferences!

4. Acknowledgements

The organizers would like to thank the speakers for the illuminating talks and the participants for their enthusiasm and energy. We would also like to thank the AIM for providing such a productive environment and particularly the AIM staff for making the organization very smooth.