Fusion categories and tensor networks

organized by
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Workshop Summary

The goal of the workshop was to bring together the two research communities of fusion categories in quantum algebraic mathematics and tensor networks in theoretical condensed matter physics. These two groups both work with the same mathematical objects, namely fusion categories, but use different language for their desired purposes. Given that both areas have seen rapid development in the past several years, it was essential for these groups to talk to one another to help further progress in both areas.

The first four mornings there were two talks, with three on Wednesday morning. For the first day, the two talks provided surveys and dictionaries between the areas of fusion categories and tensor networks. On Friday afternoon, we had two sessions of Q&A by an expert on topics that attracted a lot of attention during the week (MPO models and tensor networks and fusion 2-categories).

The speakers were:
Mon: Eric Rowell and Jutho Haegeman.
Tue: David Reutter and Xie Chen.
Wed: Yasuyuki Kawahigashi, Frank Verstraete, and Christoph Schweigert.
Thu: Fiona Burnell and Dominic Williamson.
Fri: Laurens Lootens and Theo Johnson-Freyd

Problem list and initial progress

On Monday afternoon, there was a problem session moderated by Emily Peters, at which over 30 open problems were proposed. Over the week, the conference participants worked on the following 10 problems on Tuesday-Thursday afternoons and Friday morning.

(1) Classification of braided fusion 2-categories
Following recent results on braided fusion 2-categories by Kong-Lan-Wen and Johnson-Freyd, Theo proposed that a complete classification of braided fusion 2-categories should be possible. This is in stark contrast to braided fusion 1-categories, where classification seems hopeless. The group talked about Theos strategy for solving this problem, which led to the notion of de-equivariantization for fusion 2-categories and even semisimple 2-categories. Progress was promising, and after 2 days, the group decided to continue work after the workshop and instead work on some of the other proposed problems.

(2) Explicit computations of fusion 2-categories

Recently, a definition of fusion 2-category was established by Douglas and Reutter. When working with fusion (1-)categories, it is often convenient (particularly in physical applications) to use a skeletonized version defined via explicit bases and associator isomorphisms given as tensors. What data is required for a similar description of a fusion 2-category, and how can it be used?

(3) Defining Frobenius-Perron dimensions and fusion ring of fusion 2-categories
The newly introduced fusion 2-categories of Douglas-Reutter behave significantly differently than fusion categories. In particular, Schurs lemma fails for objects, so the notions of fusion ring and FP dimension are not yet well understood. The group discussed these problems for 2 working sessions, and made progress on the notion of FP dimension of an object, which should be an algebra object rather than a number.

(4) Explicit MPO representations arising from Uq(Sl(2))
Recently it was shown that the underlying structure of the most general type of symmetries in tensor networks, called matrix product operator (MPO) symmetries, is given by a bimodule category. One prominent example are tensors with SU(2) symmetry, particularly relevant for tensor networks describing spin systems, and involves Rep(SU(2)) and Vec(SU(2)) as tensor categories with Vec as module categories containing the usual 6j and 3j symbols. For several problems, most of them arising from the study of integrability, quantum deformations Uq(sl(2)) of SU(2) play a central role, the famous 6-vertex model being the most prominent one. These problems admit a simple tensor network representation, but the precise nature of their MPO symmetries has remained elusive due to complications arising from module categories of Uq(sl(2)). In this workshop, we made progress in identifying the nature of these difficulties and were able to relate them to the mathematics literature, most notably the work on tilting modules.

(5) Higher category approach to fractons and (defect) TQFTs
In many cases, (3+1)D fracton phases can be described in terms of a defect TQFT [arXiv:2002.05166], with 3-cells labeled by a (3+1)D TQFT, and lower cells by topological defects of the relevant codimension. How can the fracton physics be seen in terms of the input fusion 2-categorical data, and what new models can be constructed from this perspective?

(6) How to see invertibility of a bimodule from the F-symbols?
An important requirement in the description of matrix product operator (MPO) symmetries in tensor networks in terms of bimodule categories is whether or not a given bimodule category is invertible. Since the input to the tensor network construction is given by the F-symbols of some bimodule category, one might wonder whether there exists a simple requirement on these F-symbols that allows us to tell whether it is invertible. Such a relation has been conjectured to exist, and a crude proof has been constructed using the tensor network language. Although some work is still required, from preliminary discussions during the workshop it appears this proof can be formalized using more standard techniques in tensor categories. We will continue working on this problem, and we believe it will provide a deep relation between tensor network concepts and tensor category theory.

(7) Construction of 3 and 4 object bicategories from 2 object bicategories for MPOs

At first sight, the construction of tensor networks for the most general type of boundaries and domain wall structures in string-net models appears to require the recoupling theory of 3 and 4 object bicategories. For all physically observable phenomena however, it seems that one can get away with just combining 2 object bicategories in some appropriate way. The deeper underlying question is whether or not a generic 3 or 4 object bicategory can be constructed by composing 2-object bicategories using the relative Deligne product. In discussing this problem with a mathematician audience, it was pointed out to us that this problem has received recent attention when considering fusion of domain walls in string-nets, and that there are cohomological obstructions to writing down F-symbols for 3 and 4 object bicategories. Building on this, we will investigate this further, and hope to be able to continue to collaborate on this problem.

(8) (2+1)d PEPS gapped => gapped boundary (lagrangian subgroup)?
The question is whether we can prove that a (MPO injective) PEPS which has a gapped parent Hamiltonian on the torus has a gapped parent Hamiltonian on the cylinder when the canonical/natural boundary condition is chosen by promoting dangling virtual indices of the PEPS at the edge of the cylinder into physical indices. If the PEPS has topological order, i.e. nontrivially MPO-injective, then this boundary corresponds to the canonical gapped boundary induced by condensing a Lagrangian algebra that is formed by excitations that are local in the chosen representation.

(9) Symmetric monoidal 2-category of quantum spin chains
C* algebras form a 2 category where bimodules are 1-morphisms and intertwiners are 2-morphisms. This observation is extremely suggestive, especially in the context of quantum spin chains, which resemble C* algebras, but also come with the additional data of a hamiltonian. Thus, it is tempting to conjecture that quantum spin chains form a 2 category with boundary defects as 1-morphisms, and MPOs (!?) as 2 morphisms. After considerable discussion it emerged that the problem might be better specified by dividing the class of quantum spin chains into gapped and gapless. The (ground states of) the gapped class admit an explicit representation via MPS which do (to appear) form a 2 category. The gapless models are related to CFTs which are far more intricate.

(10) Algorithms for solving pentagons
Associators in fusion categories are required to obey the pentagon axiom. When bases are chosen, this becomes a (highly overcomplete) collection of polynomial equations. Due to the large number of variables and equations, these are notoriously challenging to solve. Additionally, when there is multiplicity if the fusion ring, gauge freedom has so far precluded solution. Is there a systematic approach to solving these pentagon equations? The correct formulation of the problem was discussed and clarified as a basis for future collaboration.

Fusion Fridays

The group plans to continue weekly activities in the Sococo space for the foreseeable future in 2021-22 in preparation for the in-person 2022 AIM workshop. We have planned for an official activity, like a talk or Q&A session every other week. The first of these activities was a talk by Corey Jones about Centers and Tubes on April 2 on Friday at 8am Pacific time. A second talk by Jacob Bridgeman on The Ising model: from 2D classical to 1D quantum is scheduled for Friday April 16. We have also encouraged everyone to show up at this time on off weeks for unplanned group activities, which we hope will lead to further collaboration. To communicate with the participants, we have created a Google Group and invited them to join if they wish to receive further emails.