Geometric flows and Riemannian geometry

organized by
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Workshop Summary

Workshop summary: The workshop focused on several geometry flows, eigenvalue estimates and Riemannian manifolds with almost nonnegative curvature. The aim is to establish interconnection between these fields. People, from experts to graduate students, in all the three areas were invited.

The first two mornings of the workshop included survey lectures on these topics, with subsequent mornings having lectures on related current research. There was a problem session Monday afternoon, with the later afternoons occupied by group work on various specific problems. Very substantial progress was made in some groups, and the other groups all reported significant new ideas which seem promising as the basis for further investigation.

Group reports: *1) Collapsing and almost nonnegative curvature operator.* In this group, the following problem was studied: Given a manifold (M, g) on which a circle S^1 acts by isometries, define a family of metrics g_ϵ on M as a Cheeger deformation, i.e. the metric on the base of the Riemannian submersion $(M, g) \times \epsilon S^1 \to M \times_{S^1} (\epsilon S^1)$. The question is if under such a deformation the curvature operator has a lower bound independent of ϵ. Such questions arise in John Lott's work on collapse of manifolds with curvature operator bounded from below.

If the circle acts almost freely, or if the fixed point set has codimension 2, this is clearly true since the quotient is an orbifold, resp. a smooth manifold with boundary. The computations indicate that if the codimension is bigger than 2, an eigenvalue of the curvature operator goes to $-\infty$. We verifies that this is true if S^1 acts isometrically on flat R^n with $n \geq 4$, which should imply, via the action of S^1 on the space normal to the fixed point set, that this is true in general.

2) Ricci flow and convergence of almost nonnegative curvature operator. The group started considering the following question:

Let (M_i^n, g_i) be a sequence of compact n-dimensional Riemannian manifolds with $Rm_{g_i} \geq -\varepsilon_i \to 0$ and for all $p_i \in M_i$ we assume the Gromov-Hausdorff convergence $(M_i, p_i) \to \mathbb{R}^n$. Does there exist a uniform time T and a solution of the Ricci flow $g(t)$, $t \in [0, T)$, starting at (M_i, g_i)?

The answer turned out to be an easy consequence of Perelman's pseudolocality theorem. After realizing of that, we used an argument involving bounds on the heat kernel together with an upper bound on the total scalar curvature by Petrunin which gives convincing evidence that the lower bound for the curvature operator persists under the Ricci flow for positive times. As a possible application, this would lead to an affirmative answer to the following problem:

Let (M_i^n, g_i) be a sequence of n-manifolds with upper diameter bound, lower volume bound and curvature operator almost non-negative. Does the manifold M_i, for all large i, admit a metric with non-negative curvature operator?

Next the group turned to the following related question, which would lead to a generalization of a celebrated theorem by Gromov and Ruh about almost flat manifolds under weaker curvature assumptions. The specific issue is:

Consider a sequence (M_i^n, g_i) of compact n-dimensional Riemannian manifolds with sectional curvature $K_{g_i} \geq -1$ and $\frac{1}{vol_{g_i}(M_i)} \int_{M_i} \|Rm_{g_i}\| \to 0$. Moreover, for all $p_i \in M_i$ we have Gromov-Hausdorff convergence of the universal covers $(\widetilde{M}_i, p_i) \to \mathbb{R}^n$. The question is: are the manifolds M_i almost flat?

The curvature is under control again by Perelman's pseudolocality, and the only problem is that the diameter could a priori explode when the metric evolves under the Ricci flow. But we exclude that possibility by combining monotonicity of Perelman's entropy with an L^1-Sobolev inequality.

Finally progress can be made towards a generalization of the above results to the situation where one only assumes a lower Ricci curvature bound.

3) Ricci flow on a convex surface in $\mathbb{R}^3$ and the Gauss curvature flow solitons. The group started by considering the Ricci flow of a metric from a convex surface in $\mathbb{R}^3$. It turns that there is a solution to this problem via Perelman's pseudo-locality. The solution is even unique. This turns out to be known by evoking a deep result of Alexanderov. This group also considered the following question. *Let Ω be a domain with small entropy $\mathcal{E}(\Omega) \geq 0$. Assume that $Vol(\Omega) = Vol(B_1)$. Is Ω close to B_1 in the Hausdorff sense?*

Assume the origin is the entropy point, i.e., the maximum of the functional achieved at the origin. Let u be the support function of Ω with

$$\int_{S^n} \frac{x_j}{u} = 0 \quad \text{for } j = 1, 2, \ldots, n+1.$$

There is a $\tilde{\Omega}$ and a support function v satisfies

$$\det(v_{ij} + v\delta_{ij}) = \frac{1}{u}.$$

This implies that

$$Vol^{\frac{1}{n+1}}(\Omega) Vol^{\frac{n}{n+1}}(\tilde{\Omega}) \leq C_n Vol(B_1).$$

for some constant $C_n > 0$.

By a stability type argument, we also have

$$\mathcal{E}_1(\Omega) \geq \frac{1}{n+1} \log \left(\frac{Vol(\Omega)}{Vol(\tilde{\Omega})} \right)^n.$$

Define

$$\delta_{BM}(K, M) = \inf\{\lambda \geq 1, K \subset T(M) \leq \lambda K, T \in GL(n+1, \mathbb{R})\}.$$

Let Ω^* be the dual domain of Ω. If $Vol(\Omega^*) \geq 1 - \epsilon$ for some $\epsilon > 0$ sufficiently small, then

$$\delta_{BM}(\Omega, B_1) \leq 1 + \gamma \epsilon^{\frac{1}{6n}} (-\log(\epsilon))^{\frac{1}{6}}.$$

It follows that $T^{-1}B_1 \subset \Omega \subset (1+\epsilon)T^{-1}B_1$. Note that $T^{-1}B_1$ is an ellipsoid and if $\mathcal{E}(T^{-1}B_1)$ is close to 1, then $T^{-1}B_1$ must close to B_1.

The group also considered the question: *Let u be a solution of the Gaussian soliton equation which is close to* 1 *in C^0 topology. Is u close to* 1 *in the C^2 topology?*

Invoking a calculation in the paper of Guan and Ni,

$$Lu \geq KHu - u - n(n-1)K$$

where L is the linearized operator for the Gaussian soliton. As u is close to 1 in C^0 topology, the Gaussian curvature K is close to 1. Thus u is close to 1 in the C^2 topology.

4) Eigenvalue, curvature flow and the stability of the $\mathcal{W}$-entropy. In prior work, Colding-Minicozzi defined a notion of entropy stability for self-similar shrinking solutions to the mean curvature flow. Our group's motivation was to attempt to find an analogous notion of entropy stability for self-similar shrinking solutions to the Ricci flow. Corresponding versions of their stability have been developed for other geometric flows, including the Yang-Mills and harmonic map flows, and it is natural to hope that such a notion exists as well for the Ricci flow. We first sought to determine whether the stability of Perelman's μ-functional could be related to natural linear stability operators associated to his entropy functional $\mathcal{W}$. Here, recall that $\mathcal{W}$ and μ are defined by

$$\mathcal{W}(g,f,\tau) = \int_M \left[\tau(R + |\nabla f|^2) + f - n\right] \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}} dV_g,$$

and

$$\mu(g) = \inf_{\left\{f,\tau \mid \int_M \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}} = 1\right\}} \mathcal{W}(g,f,\tau).$$

Such a relationship exists among the analogous quantities, namely, λ and $\mathcal{F}$, in the work of Colding-Minicozzi. In this direction, we carried out a number of exploratory computations related to the first and second variations of the $\mathcal{W}$-functional at a shrinking Ricci soliton.

5) Level set mean curvature flow on sin-*curve and singularity formation around given analytic simple curves.* This group studied if the level set mean curvature flow can smoothen the sin-curve and if any given simple curve can be the singularity of the mean curvature for some given data. The group did not make much progresses on neither of these problems.

Summary: Many new problems are initiated during the workshop. Some progresses have been made for part of the problems. Interactions between various fields have been established. Further substantial progresses on these problem is emerging at the horizon.