Postcritically finite maps in complex and arithmetic dynamics

organized by
Benjamin Hutz, Patrick Ingram, Sarah Koch, and Tom Tucker

Workshop Summary

This workshop was devoted to the study of postcritically finite rational maps (PCF maps) from both the complex dynamic and arithmetic dynamic perspectives. The topic of PCF maps is a topic that has a significant interplay between the two areas. By bringing together researchers from both complex and arithmetic dynamics, were able to make progress on several specific problems and bring forth many new ideas in this area.

The workshop was run in the usual AIM style with typically two talks in the morning and working groups in the afternoon. The first day had talks from Joseph Silverman and Sarah Koch in the morning giving introductions to the arithmetic dynamic and complex dynamic sides of PCF maps respectively. That entire afternoon was devoted to an extensive problem discussion, where a list of approximately 20 open problems were described. These problems varied widely in their specificity, ranging from specific conjectures to broad questions. On Tuesday morning, talks were given by Laura DeMarco and Rafe Jones. Laura DeMarco discussed critical orbit relations focusing on a particular conjecture of hers. Rafe Jones discussed Galois groups of iterated maps focusing on several specific questions of his. That afternoon the organizers proposed 5 problems from the list created on Monday to work on in groups, to be discussed below. Wednesday morning the schedule was inverted and group work continued in the morning and talks were given in the afternoon. Thomas Gauthier talked about recent joint work with Charles Favre on the equidistribution of PCF maps to a natural measure in moduli space, followed by Adam Epstein talking about *Thurston rigidity*, a corollary of which is the statement that given a ramification portrait of degree $d \geq 2$ (not of Lattès type), the locus in the moduli space of rational maps of degree d which satisfies this portrait is 1) finite, and 2) reduced as a scheme. On Thursday Rob Benedetto gave a talk on sufficiently attracting fixed points in p-adic dynamics and Bob Rumley spoke on a construction of his involving Berkovich space. There were no talks on Friday.

The research groups were very active and we list some of the notable topics below.

- *Ramfication in pre-image towers:* Given a PCF map $f : \mathbb{P}^1(K) \to \mathbb{P}^1(K)$ and a fixed $\alpha \in K$ define $K_n = K(f^{-n}(\alpha))$. It is known that there is a finite set of primes outside of which the entire tower of extensions K_n/K is unramified. Rafe Jones asked whether or not this could be extended to maps of other varieties, and one of the research groups proved the same result with $\mathbb{P}^1$ replaced by an arbitrary projective variety, so long as f is assumed to be a finite PCF morphism.
- *Critical Height Conjecture:* We can define a critical height function as

$$hhat_{crit} = \sum_{c \in crit(f)} hhat(c).$$

The moduli space of degree d rational maps on $\mathbb{P}^1$ can be embedded into a projective space $\mathbb{P}^M$. This embedding induces a height function h_M. The conjecture asks for constants $c_1, c_2 > 0$ such that

$$c_1 h_M(f) - c_2 \leq \ hhat_{crit}(f).$$

There is a similar upper bound, which is known to be true. This conjecture has been proven by Ingram for polynomial maps, but is still unknown for rational maps. The group explored several approaches to this problem.

- *Notions of PCF for $\mathbb{P}^N$:* Given a map $f : \mathbb{P}^N \to \mathbb{P}^N$ there is a notion of PCF which is essentially that the forward orbit of the critical locus is a proper algebraic subvariety of $\mathbb{P}^N$. Another notion is to additionally require that the restriction of (an appropriate iterate of) f to every periodic component of the postcritical locus is also PCF, and this continues inductively at every level, leading to the notion of postcritically finite 'all the way down.' The group examined whether these two notions are equivalent. One member of the group found an example of a map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ which is postcritically finite in a certain sense, but it is not postcritically finite 'all the way down.' However, the map is not a morphism, so the definition of 'postcritically finite' in this case may require some extra care. Another group member had an interesting method of constructing examples of endomorphisms on $\mathbb{P}^n$ by taking the n^{th} symmetric product of a map $f : \mathbb{P}^1 \to \mathbb{P}^1$. The resulting map $F : \mathbb{P}^n \to \mathbb{P}^n$ will be a morphism which inherits dynamical properties of the original $f : \mathbb{P}^1 \to \mathbb{P}^1$. For example, if $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a flexible Lattès example, then the map $F : \mathbb{P}^n \to \mathbb{P}^n$ will also be a flexible Lattès example. This method of constructing examples is due to Ueda originally.
- *Critical Orbit Relations:* Let $X \subset \mathbb{A}^2$ be a curve. Is it true that there are infinitely many points $(c_1, c_2) \in X$ such that $z^2 + c_1$ and $z^2 + c_2$ are both PCF if and only if X is a vertical line, a horizontal line, or the diagonal? This was shown to be true for curves of the form $y = g(x)$ and relied on showing that the Mandelbrot set is not a Julia set for $f(z) = z^2 + c$. Work continues on more general curves.

There were also a number of impromptu working groups that formed over the course of the week as problems caught the interest of various people arising from discussions. Many of these groups made progress on the problems and are continuing work.