Engel structures

organized by
Yakov Eliashberg, Dieter Kotschick, Emmy Murphy, and Thomas Vogel

Workshop Summary

1. Motivation and background

There are only four kinds of distributions on manifolds whose germs are topologically stable: line fields (in any dimension), contact structures on odd-dimensional manifolds, even contact structures on even-dimensional manifolds, and Engel structures on four-manifolds. Of these, line fields (i.e. dynamical systems) and contact structures have been studied extensively. Even contact structures have been known for a long time to satisfy an h-principle; consequently they are of little interest geometrically as their classification is given by topological invariants. By comparison, very little is known about Engel structures.

An Engel structure is a completely non-integrable rank 2 distribution $\mathcal{D}$ on a 4-manifold M. This means that $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$ is an even contact structure, i.e. $[\mathcal{E}, \mathcal{E}] = TM$. One can show that there is a so-called characteristic foliation $\mathcal{W}$ of rank 1 which depends only on $\mathcal{E}$ and which is contained in $\mathcal{D}$. The holonomy of $\mathcal{W}$ preserves $\mathcal{E}$.

The orientability of $\mathcal{W}$ is equivalent to the orientability of M; if we assume that M and the Engel distribution are oriented, then the flag $\mathcal{W} \subset \mathcal{D} \subset \mathcal{E} \subset TM$ defines a trivialization of the tangent bundle TM, so that M has to be parallelizable.

A flow tangent to the characteristic line field $\mathcal{W}$ preserves the rank 3 distribution $\mathcal{E}$, and rotates the plane field $\mathcal{D}$ around the axis $\mathcal{W}$ in $\mathcal{E}$. On a transversal to $\mathcal{W}$, the even contact structure $\mathcal{E}$ induces a contact structure with a preferred Legendrian line field given by the restriction of the Engel plane field $\mathcal{D}$. This Legendrian line field rotates in the contact plane under the holonomy of $\mathcal{W}$.

2. The workshop

The main purpose of the AIM workshop was to review some recent progress on the existence of Engel structures and to stimulate further research by formulating geometrically interesting questions and by making connections to the modern theory of four-manifolds. For this purpose we brought together people from different areas of expertise: those few responsible for previous work on Engel structures, experts in contact topology and related topics, and experts on four-manifolds.

Following the AIM model, the workshop consisted of more or less formal lectures in the mornings, and discussions of open problems and working groups devoted to specific questions in the afternoons.

The introductory lectures on Monday morning, given by T. Vogel and R. Casals, reviewed the background on Engel structures, and discussed the basic existence results of Vogel and of Casals, Pérez, del Pino and Presas. These two lectures, and some of the discussions during the Problem Session on Monday afternoon, moderated by R. Bryant, brought into focus the extension problem for Engel structures: given a germ of an Engel structure along $S^3 \subset^4$, does there always exist an extension to an Engel structure on the ball B^4? A positive solution to this problem, and to its parametric version, would yield complete Engel flexibility, implying that Engel structures can be classified in terms of algebraic topology.

This set the stage for extensive discussions of flexibility during the whole workshop, developing several strands of ideas that different combinations of participants had started to work on before the workshop. Flexibility was also the subject of lectures by E. Murphy and by A. del Pino on Tuesday and again on Friday. On Friday morning del Pino presented his solution, together with T. Vogel, to the non-parametric extension problem using the notion of an Engel Lutz twist. Other approaches to the extension problem, especially in its parametric version, perhaps only for special classes of Engel structures, are actively pursued by several participants, but did not quite come to fruition during the week of the workshop.

In his Tuesday lecture, del Pino proposed a notion of overtwistedness for Engel structures. Unlike overtwistedness in contact geometry, which can be localized on an overtwisted disk, this notion of overtwistedness for Engel structures is a global condition formulated in terms of the Engel plane making (several) full turns along every leaf of a foliation of rank 1 which is tangent to $\mathcal{D}$ but never tangent to $\mathcal{W}$. The implications of this notion, and alternative, more localized definitions of overtwistedness, are the subject of ongoing research by some of the participants.

The lecture by D. Kotschick, based on joint work with T. Vogel, discussed bi-Engel structures, which are pairs of Engel structures inducing the same even contact structure, but turning in opposite directions without ever coinciding. For such Engel structures there are never any full turns of the Engel planes along leaves of the characteristic foliation, so they are not obviously overtwisted in the sense described above. Engel structures coming from bi-Engel structures are, for the time being, the most promising candidates for Engel structures which are not overtwisted. However, for all definitions of overtwistedness which proposed so far it is very difficult to show that the required conditions are not satisfied in a given Engel structure.

Bi-Engel structures correspond to even contact structures whose characteristic foliations satisfy an appropriate notion of hyperbolicity. This is just one of several hints of strong connections between Engel geometry and the theory of dynamical systems. Such connections were the subject of an active working group that met every afternoon of the workshop. One of the problems discussed in this group concerned the possible constraints on the topology of the canonical one-dimensional characteristic foliation on an Engel manifold, i. e. given an even contact structure on a four-manifold, under what conditions on the characteristic foliation can one can inscribe an Engel structure in it? The problem is analogous to the problem in three-dimensional contact geometry concerned with the existence of a contact structure tangent to a given one-dimensional foliation. Some progress was achieved in finding separately necessary and sufficient conditions. The participants of the discussion group plan to explore this subject further.

The lecture by V. Colin, based on joint work with F. Presas and T. Vogel, discussed Engel structures adapted to open book decompositions of parallelizable four-manifolds. Related cut-and-paste constructions were discussed in some of the afternoon working groups, focusing for example on transverse tori and their applications to branched coverings of Engel structures.

The lectures by R. Montgomery and by R. Bryant provided an outlook into the zoo of distributions in higher dimensions, beyond Engel plane fields in dimension four. In particular, both lectures discussed Goursat structures in arbitrary dimensions. This formed the background for some of the work of an afternoon group devoted to Engel cobordism and concordance. It turned out that the structure required on a five-manifold in order to have a meaningful notion of Engel cobordism or concordance is more than just a contact structure, and is closely related to Goursat structures.

Another afternoon working group discussed (putative) connections between Engel structures and complex geometry. One of the outcomes was an example produced by R. Bryant, of an Engel structure on a complex torus, for which the Engel plane field is a complex subbundle of the tangent bundle. In addition, the question how to characterize submanifolds of codimension two in complex manifolds of real dimension 6 such that the space of complex lines tangent to the submanifold is an Engel structure was considered. The relevant computations are lengthy and this group decided to pursue questions concerning cobordisms between Engel manifolds (see the previous paragraph).

Yet another working group investigated Engel knots in the standard Engel structure on $\mathbb{R}^4$. In this case the connected components of the space of Engel knots had been previously determined by J. Adachi (and later by H. Geiges), and the working group focused on understanding the fundamental group of this space. In particular, a loop was examined in detail which was proposed as a potential example of a non-contractible but formally contractible loop in the space of Engel knots in $\mathbb{R}^4$.

At the end of workshop activities on Friday we held another lengthy problem session, moderated by R. Bryant with the assistance of Y. Eliashberg. During this session we took stock of progress made during the week and we edited and updated the problems compiled on Monday in light of this progress.