Minimal energy problems with Riesz potentials

organized by
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Workshop Summary

Introduction

The workshop focused on problems related to the distribution of particles driven by a certain repulsive/attractive law (e.g., Coulomb potential). These problems find applications, and require tools from statistical mechanics, geometric measure theory, numerical integration, Riesz frames.

The workshop brought together several groups of people who always wanted, but did not always have an oportunity, to interact with each other. The participants of the workshop were very diverse, and were a good mix of members of underrepresented groups, junior researchers, people from institutions with and without graduate program. As a result of the workshop, several new collaborations were formed and already produced at least one preprint on arXiv.

Talks

Ten talks were delivered during the workshop. Below is a very brief summary of the talks.

Thomas Leble gave a survey of methods from statistical mechanics, and presented several open problems as well as valuable references.

Doug Hardin gave a survey of the current state of the Riesz Energy theory. The talk raised an important question that eventually formed a new research collaboration.

After a big interest the previous talked raised, Florian Thiel talked about related problems from the point of view of Material Sciences.

Carlos Beltran gave a talk on the applications of Determinantal Point Processes, which is a tool from probability theory that is, somewhat unexpectedly, very useful for the discrete Riesz energy.

Sergiy Borodachov gave a talk on the current state of the Minimal Dicrete Polarization theory.

Peter Dragnev gave a talk on the applications of linear programming, in particular on the new way to use Levenstein methods to obtain bounds on the minimal discrete energy.

Maryna Viazovska gave a talk on a particular application of linear programming and modular forms to prove universal optimality of some lattices for a wide class of energies.

Further, Peter Grabner gave a talk that involved some revelation of why modular forms and the methods presented in the previous talk work so well.

Dmitriy Bilyk talked about some phenomena related to "clustering" of the minimizers for discrete energy for certain potentials.

Laurent Betermin talked about the minimal discrete energy problems for Lennard-Jones potentials, which is known to be a tool to study crystallization phenomena.

Problem sessions The problem sessions went as expected, with every group having about the same number of participants. The main problems discussed were the following.

- **Scar behavior on the sphere**: all numerical methods to obtain a large optimal configuration for the Riesz energy on the sphere show the following phenomenon. If one takes these points and draw their Voronoi cells, most of the cells are hexagons, but some are pentagons or heptagons. The arrangement of these "outlier" always looks like a scar on the sphere. It was conjectured that understanding the behavior of these scars can lead to higher-order asymptotic terms for the energy, which is a very long-standing open problem. A new research collaboration was formed during the discussion of this problem.
- **Maximal discrete polarization**: this problem session focused on a particular problem related to an optimal configuration for five points in the unit disc. It was understood that this problem, despite it's trivial statement, is very hard; several conjectures were formed. This problem will be further tackled after the workshop. One other topic related to this problem session was discussed during lunch breaks, and eventually led to a proof of an important result that patches a hole in an old paper.
- **Discreteness of minimizers**: the "clustering" phenomenon was discussed in details. During the discussion, some known results due to Carrillo–Figalli–Patacchini were generalized. The general problem discussed during this session was known to be very hard, but with some new ideas and collaborations it might get solved sooner.
- **Determinantal point processes on general manifolds**: following the applications of the DPP's on the sphere, this problem session focused on generalizing these applications to general manifolds. Some preliminary results were obtained during the session, and there is a hope that they will lead to more general results. One particular positive result obtained was that the random points distributed according to the particular DPP will almost surely converge, in the weak sense, to the volume measure of the manifold.
- **Determinantal point processes on some particular manifolds**: the group discussed building a DPP on projective spaces to apply them to the minimal discrete energy on these spaces. The group reported that they came close to obtaining an interesting result similar to the one already obtained on the manifold $SO(3)$.
- **Linear programming**: the group participating in this discussion tried to push the ideas from linear programming to obtain new results, in particular, for the hexagonal lattice in $\mathbb{R}^2$ and the lattice D_4 in $\mathbb{R}^4$. Although these problems are known to be extremely difficult, with some new ideas there is a hope to get new important results in this direction.