Double ramification cycles and integrable systems

organized by
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Workshop Summary

The double ramification (DR) cycle has given rise to a number of connections between various mathematical areas: intersection theory of moduli spaces of curves, integrable systems, logarithmic/tropical geometry, and abelian varieties. This workshop brought together representatives from these different communities, with the goal of fostering interactions that benefit from the different points of view on the subject. The main topics for the workshop were:

- DR vs. Dubrovin-Zhang hierarchies equivalence conjecture.
- Generalizations of CohFTs and Givental-Teleman classification.
- DR cycles and admissible covers cycles.
- DR cycles arising from logarithmic compactifications of the universal Jacobian.

The workshop followed a typical AIM workshop format. Every morning there were two talks: the first two days the speakers were asked to provide a broad background and an overview of the subject area; one of the main goals that these talks successfully achieved was to establish common ground among all participants. The third and fourth days were devoted to more specialized talks, focusing on recent developments. These talks were vastly beneficial for the afternoon groupwork. Friday morning we invited two graduate student participants to talk about their current research projects: this provided a good opportunity for vertical integration among workshop participants.

Monday afternoon hosted a discussion on open problems, moderated by Dmitry Zakharov. Many different participants proposed questions, and the ensuing discussion helped refine such questions into concrete research problems. Zakharov made a point of summarizing each discussion on the board. Andreas Gross and Johannes Schmitt acted as scribes for the moderated problem session. By monday evening, they shared with the organizers a list of fifteen different problems. The organizers selected eight problems to bring to everyone's attention on tuesday. After a brief discussion and a round of voting four working groups were formed. The remaining time on tuesday and on all subsequent afternoons the groups worked on their problems. An interesting feature is that many participants were involved in the work of more than one group, splitting their time between different rooms. The afternoons still felt like there was a substantial amount of interactions among all participants.

At the beginning of friday afternoon about one hour was spent on an extended summary of each group's work. The outcomes of these working groups are summarized below.

Working group projects

DR hierarchy and KP hierarchy.
Participants: Younghan Bae, Xavier Blot, Olivia Dumitrescu, Paolo Rossi, Dimitri Zvonkine

In a paper of Pandharipande, Pixton and Zvonkine it is remarked that Witten's r-spin cohomological field theory is polynomial in r, for r big enough. P. Rossi proposed to study the $r = 0$ term of such polynomial CohFT. The definition actually requires some care, but one ends up with an infinite rank partial CohFT. From work of Buryak, Dubrovin, Guéré and Rossi, it is known that a DR hierarchy can be associated to a partial CohFT. P. Rossi conjectured that DR hierarchy associated to the $r = 0$ part of Witten's r-spin CohFT should be the KP hierarchy and this working group's aim was to gather evidence towards such result.

We can say that this project was successful. It was understood that the above infinite rank partial CohFT actually produces an even bigger hierarchy, inside which KP is found as a reduction. KP is accordingly associated to the restriction of the partial CohFT to a subspace of the full phase space. The result is not a partial CohFT anymore, nor it is an F-CohFT, but the relevant intersection numbers behave like it was. The first few equations were computed and the KP hierarchy emerged from such computations.

Now it is left to prove that indeed, the full hierarchy (an infinite number of equations) coincides with KP, and to study the bigger partial CohFT and associated DR hierarchy.

DR vs Admissible covers cycles.
Participants: Renzo Cavalieri, Emily Clader, Yaim Cooper, Felix Janda, Aaron Pixton, Johannes Schmitt, Rosa Schwartz, Fenglong You, Jonathan Wise.

This group studied the difference between Double Ramification (DR) cycles and Admissible cover cycles by analyzing contributions to the DR cycle supported on non-main components of the moduli spaces of relative stable maps. Each such component supports a virtual class, that can be expressed as a Hodge polynomial. The subtle issue that arises is that different components intersect, and there may be interference among virtual classes on such intersections. With the aid of Johannes Schmitt's computer code `admcycles.sage`, the group started by gathering data with some low degree and genus computations:

- $d = 1, g \leq 4$**::** the admissible cover locus is empty in this case. The contribution to the DR cycle may be interpreted as a sum of virtual contributions from the moduli spaces of relative stable maps;
- $d = 2, g \leq 3$**::** the difference between the DR and the admissible cover cycle was computed.

Following the calculations and experiments, Aaron Pixton proposed a conjectural graph formula describing the degree one classes $DR_g(1, -1)$ as sum of strata decorated by Hodge

polynomials. After group discussions, a combinatorial framework that might allow to extend such conjecture to higher degree was developed.

Tropical DR locus.
Participants: Renzo Cavalieri, Andreas Gross, Steffen Marcus, Aaron Pixton, Dmitry Zakharov.

In recent work, Ulirsch and Zakharov defined a tropical DR locus, a pure dimensional sub-cone complex of the moduli space of tropical curves. Work in progress by Cavalieri, Gross and Markwig is aiming to give an intersection theoretic definition for tropical ψ classes on moduli spaces of tropical curves. This group focused on a combination and application of these two lines of thought. The goal is to show correspondence theorems: that the tropical DR locus may be expressed in terms of tropical ψ classes by Pixton's classical formula, and that the intersection numbers of the tropical DR locus with tropical ψ classes agree with the classical formulas of Buryak, Shadrin, Spitz and Zvonkine. During the week, the group focused on the case $g = 1, n = 2$. In this case, up to the formalization of some details, the goals appear to have been achieved. Such experiment makes it plausible that the genus one case may be approachable for general n.

DR cycle and g-th power of the Θ-divisor.
Participants: David Holmes, Felix Janda, Jesse Kass, Nicola Pagani, Aaron Pixton, Jonathan Wise.

On $M^{ct}_{g,n}$, the class of the DR cycle is equal to $\frac{a^*\Theta^g}{g!}$, where a denotes the Abel map to the moduli space of abelian varieties, and Θ is a translate of the theta divisor. This group focused on extending such a formula to the Deligne-Mumford compactification $\overline{M}_{g,n}$. The idea is to understand an extension of the theta divisor to a compactified universal (logarithmic) Jacobian. The group investigated whether some specific translate of the theta divisor lends itself more naturally to being extended. The group made a computation in genus 2, which led them to conjecture a more general formula. However, the group realized that their formula needs further refinement before it is expected to hold in general.