Degenerations in algebraic geometry

organized by
Brian Osserman and Ravi Vakil

Workshop Summary

The workshop focused on several specific applications of degeneration techniques in algebraic geometry, including Schubert cycles in Grassmannians, curves interpolated through points in the plane, and linear series on curves. A number of tropical geometers were included in order to provide their perspective on these problems.

The first two mornings of the workshop included survey lectures on these topics, with subsequent mornings having lectures on related current research. There was a problem session Monday afternoon, with the later afternoons occupied by group work on various specific problems. Very substantial progress was made in two groups, and the other groups all reported significant new ideas which seem promising as the basis for further investigation.

Group reports

Schubert cycles associated to rational normal curves. It is known that in characteristic 0, the intersection of Schubert cycles associated to general osculating flags on a rational normal curve is transverse. However, there is no geometric proof, and such a proof would likely be helpful for generalization. This group studied the possibility of proving transversality by adapting Vakil's techniques for degenerating the flags. Although everything is at a very preliminary stage, this appears to lead to toric degenerations of the underlying projective space as well as the associated Grassmannian. We hope that this will be useful in studying the problem.

Interpolation and bounded negativity. The interpolation problem – studying curves passing through points in the plane with fixed multiplicities – is a fundamental open problem in algebraic geometry. Groups considered different aspects of the interpolation problem throughout the workshop, including bounded negativity of self-intersections, a convex bodies approach to Nagata's conjecture, and the possibility of a tropical approach. Again at a preliminary stage, the group was able to formulate some test cases for what a tropical approach should look like, and began to investigate them with promising results in a first example.

Brill-Noether theory of binary curves. Nodal curves obtained by gluing a pair of projective lines to one another repeatedly – "binary curves" – show up naturally in various contexts, including as special fibers of modular curves. This group attempted to extend work of Caporaso studying the linear series on such curves. One member of the group wrote a computer program which verified good behavior in many examples. The group also rephrased the question in terms of an intersection of Schubert cycles, and in some basic examples was able to use degenerations to prove that the cycles intersect in the expected way.

Geometry of Brill-Noether varieties. A moduli space of linear series on a general curve is a "Brill-Noether variety." In their proof that moduli spaces of curves of high genus are of general type, Eisenbud and Harris proved a formula for the genus of the Brill-Noether variety in the case that it is 1-dimensional. In very recent work, Chan, Lopez, Pflueger and Teixidor have given a new proof of this fact (and generalized it) using limit linear series. This group considered the generalization of their approach to arbitrary dimension, and made major progress, with new conjectures and a clear plan for the proof of the general case.

Higher syzygies of curve imbeddings. Green's theorem and the Green-Lazarsfeld conjecture study the syzygies of imbedded algebraic curves. This group investigated the possibility of using tropical techniques to prove new results, with a focus on degrees of generators for the ideals. The group developed an approach, and verified that it was successful in the first nontrivial case.

Stratifications of $(\mathbb{P}^1)^n$. An anticanonical divisor on $(\mathbb{P}^1)^n$ induces a stratification, and the working group studied the question of how the stratifications varied under a degeneration of a given divisor. This group was largely able to solve the problem in question, with a conjecture and the outline of a proof.

Counting curves with maximal tangency. Motivated by conjectures in mirror symmetry, this group considered the question of counting the number of rational curves in the plane maximally tangent to a fixed elliptic curve. The group spent most of the workshop developing a tropical formulation of the problem, and testing it in the first examples. So far, the approach looks promising.

Further plans

Given the initial progress, many groups plan to continue working together on their problems. Details are yet to be worked out, but the AIM SQuaREs program could be a natural fit in some cases.