The Cauchy-Riemann equations in several variables

organized by
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Workshop Summary

The workshop on the Cauchy-Riemann equations in several variables began Monday morning with a short (15 minute) talk by John D'Angelo, one of the organizers. This talk had two purposes; one was to describe the format of the workshop, and the other was to formulate a problem on homotopy for proper mappings between balls. A second organizer, Dror Varolin, then gave a careful presentation on twisted $\overline{\partial}$ methods for metrics on holomorphic line bundles over complex manifolds. The concept of functions with self-bounded gradient, first defined by McNeal, who was present at the workshop, was introduced and became one of the topics discussed in a working group. The third organizer, Bernhard Lamel, was unable to attend the workshop.

Peter Ebenfelt then gave an hour talk in which he showed how to derive the *gap conjecture* of Huang-Ji from a conjectural algebraic property of the rank of certain positive Hermitian polynomials: Suppose a squared norm of a holomorphic polynomial mapping is divisible by $||z||^2$. What constraints are there on the rank of this squared norm? Ebenfelt formulated a precise algebraic conjecture and indicated how it implied the gap conjecture. A large working group developed to work on this question.

After these introductory talks, the participants divided into several working groups. One of these groups considered estimates for the Cauchy-Riemann operator $\overline{\partial}$, another focused on the geometry of functions with self-bounded gradient, and a third considered the question posed by Ebenfelt. These groups were fluid, and people moved among them during the week.

On Tuesday morning, Charles Epstein presented a general talk on CR geometry, beginning with the Hans Lewy equation, continuing with Nirenberg's work on perturbation of three dimensional CR structures, the work of Eliashberg, Epstein-Burns and Bland-Lempert on deformations of global CR structures on the sphere, and leading to several questions on deformations of CR structures. These questions included one on the embedability of certain CR structures arising from Kuranishi's work on versal deformations of analytic singularities. After a short break, Friedrich Haslinger presented a talk on the functional analytic set-up involved in the $\overline{\partial}$-Neumann problem. This talk helped solidify the backgrounds of those involved in two of the three working groups. In particular, Haslinger discussed compactness for the $\overline{\partial}$-Neumann operator N, a topic of considerable interest to many of the participants.

On Tuesday afternoon all participants gathered to discuss these talks and to refine the topics discussed in the working groups. We then broke into working groups and intense discussion began.

Wednesday morning featured the only other lecture of the meeting. Michael Christ discussed sharp off-diagonal decay estimates for the Bergman kernel on large powers L^λ of

an ample holomorphic line bundle L over a compact complex manifold. He showed various results such as that the decay can be at most as slow as $e^{-c\sqrt{\lambda \log(\lambda)}}$, and cannot be any faster when the metrics are smooth but not real analytic, provided these metrics satisfy a certain symmetry which he explained in his talk. Christ then posed the general question regarding the necessity of this symmetry. Zelditch had conjectured that the conclusion holds even without this symmetry assumption.

Early on Wednesday afternoon various people described in more detail what was being discussed in the working groups. Straube described aspects of $\overline{\partial}$ estimates, obstructions to compactness, and the relationship between global regularity and the existence of Stein neighborhood bases. Zaitsev described some of the discussion on multi-types and constructions of plurisubharmonic functions due to Catlin nearly thirty yers ago. Catlin, who seldom attends meetings, provided considerable insight on these topics (which remain of current interest). D'Angelo mentioned a result on homotopy he and Lebl obtained on Tuesday at the workshop. This result did not answer the original question posed the first day, but it is of some interest: A term of a Whitney sequence with target dimension N is homotopic in dimension $N+1$ to the map $z \to (z, 0)$. McNeal provided additional information on functions with self-bounded gradients. Ebenfelt mentioned some of the ideas from his working group, including the use of the Hilbert function from commutative algebra in the original problem about ranks of multiples of $||z||^2$.

On Thursday we held a problem session, moderated by Charlie Epstein and scribed by Yunus Zeytuncu. The participants generated a coherent list of fifteen difficult problems, most of which were closely connected to the topics discussed by the working groups. The statements of these problems evolved throughout the workshop via informal discussions between the participants and the scribe.

The participants continued working on Friday morning and the discussions had become quite lively. On Friday afternoon, the participants gathered together and presented progress made by the working groups. By this time the number of working groups had increased. Zeytuncu and Chakrabarti discussed a problem they considered regarding boundedness properties of the Bergman projection. Lebl outlined some progress on Ebenfelt's conjecture, which was considered in the case of monomial mappings. They solved a version of the question in the monomial case, where one can gain insight from certain Newton diagrams. Harrington outlined a negative answer to a question posed by Varolin regarding certain constants associated to the Bergman kernel of a strictly pseudoconvex domain, thereby solving one of the problems posed at the beginning of the workshop in the self-bounded gradient group. Kossovski presented progress on the work of the groups studying problems in CR geometry, and posed a number of questions regarding formal and convergent embedability of real analytic CR structures into certain model CR manifolds of higher dimensions. Finally, Sorin Dragomir reported on discussions he had with Francine Meylan regarding local embedability of deformations of a certain CR structure that arises, via a theorem of Hill, Lawandowski and Nurowski, from the Gödel solution of the Einstein field equations. The workshop then ended with another hour of final discussions in various groups.

One of the primary purposes of the meeting was to focus on the link between the Bergman kernel and estimates for $\overline{\partial}$. This topic was a significant part of the discussions of two of the working groups. Curvature and potential-theoretic considerations were mentioned in the original announcement, and these topics were also significant parts of the workshop.

The organizers expect that the collegial and stimulating environment of this AIM workshop will lead to additional progress.