Analytic combinatorics in several variables

organized by
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Workshop Summary

The workshop was held 4-8 April 2022 at AIM in San Jose, on the topic of Analytic Combinatorics in Several Variables (a subject loosely delineated by the book of R. Pemantle and M. C. Wilson, now being revised for a 2nd edition with additional author S. Melczer).

Introductory lectures were given by (Monday) Robin Pemantle, Stephen Melczer, Yuliy Baryshnikov, (Tuesday) Mark C. Wilson and Terrence George, and an invited lecture on Wednesday was given by Alexander Kolpakov. A problem discussion on Monday afternoon led to four main areas of enquiry. Semi-independent reports from each of the groups studying these are included below.

Amoebas.

The (somewhat virtual) group dealing with amoeba topology was initially looking into the (20 years old) problem of the "natural cycles independence."

Namely, consider a Laurent polynomial P in d variables, and the corresponding amoeba $\mathrm{amoeba}_P \subset \mathbb{R}^d$. To each component of the complement to the amoeba (an open convex subset of $\mathbb{R}^d$ one can associate a d-homology class (a natural class) in $H_d(\mathbb{C}_*^d - \{P = 0\})$: the class of the tori-preimages of the points in that component. The problem is to prove that these classes are independent in $H_d(\mathbb{C}_*^d - \{P = 0\})$.

It seems that we can solve this problem at least for *generic* Laurent polynomials, and have a clear line of attack in the general case.

During the workshop, it also became clear that the standard tool of singularity theory — deformation of an object avoiding the discriminant — can help us with other outstanding problems, where the topology of the pole variety (of a generating function) matters: in particular the notorious "multiplicity 3" problem that arose in our study of lacunary families of generating functions.

Among smaller problems (intended as a warm-up), the question that would help determine the asymptotics of the number of vertices of Stokes polytopes in the simplest nontrivial family was quickly solved by M. Drmota and S. Melczer.

Algebraic generating functions.

The ACSV project has extensive results for asymptotic coefficient extraction from rational generating functions in fixed dimension. Algebraic functions that are not rational occur in many natural problems. In the univariate case, there is extensive effective asymptotic theory (e.g. Flajolet-Sedgewick book, results of Drmota and coauthors). In general dimension, Torin Greenwood derived asymptotics for special algebraic singularities, with very considerable work, using custom contours. The main question for the workshop is: can we instead treat algebraic functions by "embedding" them into rational ones, and then using rational ACSV? In other words, given algebraic f, can we find a higher-dimensional R such that f occurs as a "diagonal" of R?

Let P be the minimal polynomial of f. A result of Furstenberg gives an explicit formula for such an R when P has only a single root passing through the origin. The utility of this for asymptotics has recently been shown by Greenwood, Melczer, Ruza and Wilson (FPSAC 2022). Several known methods reduce the general case to this case and thereby solve the problem. In the workshop we clarified these methods (Safonov, Denef-Lipschitz, Adamczewski-Bell), showed that they give different answers in general, and realized that although the third is dominated by the first, the first is algorithmic and the second is not, the second allows more scope for inspired choices leading to "good" embeddings, namely those that allow existing ACSV methods to work well. This is easily seen in the example $x/\sqrt{1-x}$. The rational functions R arising often have interesting tricky points of analysis, such as critical points at infinity, which connects the work of this group to that of one of the other groups.

The group also started analysis of the variety of rational functions with zero diagonal and having given degree denominator, and plans a systematic generation of embeddings for interesting combinatorial examples. We also formalized the concept of a weakly $\mathbb{N}$-rational GF, and conjecture that every $\mathbb{N}$-algebraic GF embeds into a weakly $\mathbb{N}$-rational GF. This relates to work of another of the groups.

Generalizing $\mathbb{N}$-rational and $\mathbb{N}$-algebraic GFs.

We introduce a new class of generating functions (GF) we call $\mathbb{T}$-*Algebraic*, which generalize both the $\mathbb{N}$-Rational and $\mathbb{N}$-Algebraic GFs, and is a subclass of $\mathcal{D}$-Algebraic GFs.

We show that $\mathbb{T}$-Algebraic GFs is a rich class, and in particular includes a family of GFs enumerating maps with various properties (triangulations, 2-connected, bipartite, etc.), closed formulas for which were famously obtained by W.T. Tutte. In the earlier work by Banderier and Drmota, this family was proved to be *not* $\mathbb{N}$-Algebraic using asymptotic analysis.

In a different direction, we prove that $\mathbb{T}$-Algebraic GFs have an analytic continuation. This implies that GFs for partitions $P(t)$ and plane partitions $P_2(t)$ are *not* $\mathbb{T}$-Algebraic. This is notable since $P(t)$ is $\mathcal{D}$-Algebraic, while for $P_2(t)$ whether this GF is $\mathcal{D}$-Algebraic remains an open problem.

Critical points at infinity.

In ACSV, we are interested in studying asymptotics of the coefficients of multivariate generating functions of the form $F(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ $(\mathbf{z} = (z_1, \ldots, z_d), \mathbf{r} = (r_1, \ldots, r_d))$, where frequently in applications, $F(\mathbf{z}) = \frac{P(\mathbf{z})}{Q(\mathbf{z})}$ is a rational function.

Let $V(Q) = \{\mathbf{z} \in (\mathbb{C}^\times)^d : Q(\mathbf{z}) = 0\}$ denote the hypersurface in $(\mathbb{C}^\times)^d$ defined by the vanishing of Q. Let $h_{\mathbf{r}}(\mathbf{z}) = -\mathbf{r} \cdot \log|\mathbf{z}|$ be the *height function*, where $\log|\mathbf{z}| = (\log|z_1|, \ldots, \log|z_d|)$.

The procedure to determine asymptotics is as follows:

(1) Use the Cauchy integral formula

$$a_{\mathbf{r}} = \frac{1}{(2\pi i)^d} \int_T \mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}},$$

where $T \subset (\mathbb{C}^\times)^d$ is a (real) torus of integration contained in the domain of convergence of $F(\mathbf{z})$.

(2) Deform the torus T to a torus T' contained in a domain of convergence of $F(\mathbf{z})$ where the height function is unbounded from below. During the deformation, we intersect the variety $V(Q)$ in a $(d-1)$-cycle $I(T, T')$. Taking a multivariate residue, the integral (1) becomes an integral of a $(d-1)$-form over $I(T, T')$.

(3) Deform the cycle $I(T, T')$ in the direction of decreasing $h_{\mathbf{r}}$ until it reaches critical points of $V(Q)$ i.e. points $p \in V(Q)$ where $dh\big|_{V(Q)}(p) = 0$ or equivalently the matrix

$$\begin{bmatrix} r_1 & \cdots & r_d \\ z_1 \frac{\partial Q}{\partial z_1}(p) & \cdots & z_d \frac{\partial Q}{\partial z_d}(p) \end{bmatrix}$$

has rank ≤ 1, i.e. all of its 2×2 minors vanish. Since this condition does not change upon scaling $\mathbf{r}$, we will regard $\mathbf{r}$ as an element of $\mathbb{CP}^{d-1}$ below.

(4) Perform saddle point integrals at the critical points.

Our focus is on step 3, and eventually step 4. Since the variety $V(Q)$ is not compact, we might have a situation where the cycle $I(T, T')$ can be deformed to infinity while $h_{\mathbf{r}}$ approaches a finite value. To deal with this issue, [BMP] compactified $V(Q)$ by taking its closure in $\mathbb{CP}^d$, and defined a *critical point at infinity* (CPAI) to be a pair $(p, \mathbf{r})$ where $p \in \overline{V(Q)} - V(Q), \mathbf{r} \in \mathbb{CP}^{d-1}$ such that there exists a sequence $(p_n, \mathbf{r}_n)$ $(p_n \in V(Q), \mathbf{r}_n \in \mathbb{CP}^{d-1})$ where p_n is an $\mathbf{r}_n$-critical point in $V(Q)$. With this definition, [BMP] provided a checkable condition which allows the behavior at infinity to be ignored.

However the compactification of $V(Q)$ described above has some drawbacks:

(1) $\overline{V(Q)}$ often has undesirable singularities due to points at infinity getting collapsed.

(2) The height function $h_{\mathbf{r}}$ does not extend continuously to the points at infinity.

(3) The definition of CPAI involves a limit and we would like an algebraic definition like (3) above.

(4) Optimistically, we would like to perform ACSV step 4 and obtain asymptotics even in the presence of CPAI, but the drawbacks listed above make this difficult.

A compactification of $V(Q)$ that is better behaved is obtained by taking its closure $\overline{V(Q)}$ in the toric variety X_N of the Newton polytope N of Q. The toric variety X_N has a stratification

$$X_N = \bigsqcup_{\text{faces } \Gamma \text{ of } N} X_\Gamma^\circ,$$

where each stratum $X_\Gamma^\circ \cong (\mathbb{C}^\times)^{\dim \Gamma}$ is a torus. The construction of the toric variety X_N is adapted to $V(Q)$ which deals with drawback 1. For drawback 2, while the height function $h_{\mathbf{r}}$ does not extend to all of X_N, it does extend to the orbits X_Γ° such that $\mathbf{r}$ is parallel to Γ, and as we state in the dream theorem below, these are the only orbits where we expect to see CPAI in the direction $\mathbf{r}$.

During the AIM workshop, we identified the theorem we would eventually like to prove (in particular, handling drawbacks 3 and 4 as well). We state it now and then discuss the progress we have made during the workshop towards this goal.

Theorem 0.1 (Dream)**.** *Suppose* $(p_n, \mathbf{r}_n)$ *converges to a CPAI in the direction* $\mathbf{r}$*. Let* $p \in X_N$ *be the limit of* p_n *in* X_N*. Then*

(1) $p \in X_\Gamma^\circ$*, where* Γ *is a face of* N *parallel to* $\mathbf{r}$*.*
(2) p *is a critical point (in the sense of 3) of the continuous extension of the height function* $h_\mathbf{r}$ *to* X_Γ° *.*
(3) $h_{\mathbf{r}_n}(p_n) \to h_\mathbf{r}(p)$*.*
(4) *After deforming the chain* $I(T, T')$ *to* p*, we have an integral that can be handled by ACSV techniques.*

Item 2 in the dream theorem provides the algebraic definition of CPAI in drawback 3, and item 4 is our hope that after disposing with drawbacks 1, 2 and 3, handling drawback 4 becomes accessible.

When Γ is a facet of N, and $p \in X_\Gamma^\circ$, we know how to prove items 1, 2 and 3. In particular, the dream theorem is true when $d = 2$. During the workshop, we attempted item 4 in two examples:

(1) We first tried the simplest example where $F(\mathbf{z}) = \frac{1}{1-z_1-z_2}$. We could take another residue at the critical point at infinity to derive asymptotics.
(2) We then tried the generating function in [**?**, Example 2], but have not yet finished this computation because of difficulties in identifying the cycle $I(T, T')$.

The main issue in proving the dream theorem in general is that CPAI are defined using a limit in the analytic topology in [BMP], while the construction of toric varieties is algebraic. For Γ a facet of N, we have a chart $(\mathbb{C}^\times)^d \sqcup X_\Gamma^\circ \cong \mathbb{C} \times \mathbb{C}^{d-1}$, which gives the analytic structure. When Γ is not a facet, points in the orbits $\overline{X_\Gamma^\circ}$ might be singular and the analytic structure is unclear. Yuliy Baryshnikov suggested looking at [BGL] which contains an analytic construction of toric varietes.

Bibliography

[BMP] Yuliy Baryshnikov, Stephen Melczer, and Robin Pemantle. Stationary points at infinity for analytic combinatorics. *Foundations of Computational Mathematics*, pages 1–34, 07 2021.

[BGL] Dan Burns, Victor Guillemin, and Eugene Lerman. Kähler metrics on singular toric varieties. *Pacific J. Math.*, 238(1):27–40, 2008.