Beyond Kadison-Singer: paving and consequences

organized by
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Workshop Summary

Introduction

The 1959 Kadison-Singer Problem had originated from a comment in the work of Paul A. M. Dirac, who studied the question of how our notion of classical logic and the theory of classical mechanics can be incorporated in a more fundamental theory of quantum systems.

Dirac claimed that if any state of a classical mechanical system is determined by measuring countably many quantities, where the order in which the measurements are performed does not have any effect on the outcomes, then there is essentially only one way to realize these states in the standard formulation of quantum mechanics [Dirac].

In 1959, Richard Kadison and Isadore M. Singer had formulated this claim as an open problem in the theory of von Neumann algebras [KS]: If the classical observables are given by the self-adjoint elements in the abelian von Neumann algebra of bounded multipliers on the Hilbert space of square-summable infinite sequences $\ell^2(\mathbb{N})$, then does any state on this von Neumann algebra extend uniquely to a state on the quantum observables, the set of all bounded self-adjoint operators on this Hilbert space?

Kadison and Singer believed "that such extension is non-unique", perhaps because they could show that for other abelian subalgebras of the von Neumann algebra of bounded operators on the Hilbert space $\ell^2(\mathbb{N})$, the extension was non-unique [KS]. Nevertheless, after more than 50 years, Dirac's claim had resisted all attempts to prove or disprove it.

Over 55 years it was shown that the Kadison-Singer Problem was equivalent to fundamental open problems in a dozen areas of research in both pure and applied mathematics, including the Anderson Paving Conjecture, the Bourgain-Tzafriri Conjecture, Akemann-Anderson projection paving, the Feichtinger Conjecture, and Weaver's paving conjecture, see [CasazzaTremain,CEKP] and [Weaver]. The surprising solution of these equivalent problems came from a proof of Weaver's paving conjecture by Adam Marcus, Dan Spielman, and Nikhil Srivastava.

The main purpose of the workshop was to understand the technique used by Marcus, Spielman and Srivastava for the proof of Weaver's conjecture, to develop a constructive version of the proof and to explore consequences of the various affirmative formulations of the Kadison Singer problem in different fields of mathematics.

Workshop activities

The first part of the workshop was dedicated to a review of the history of the Kadinson Singer problem and its reformulations as well as an overview of the techniques that entered its proof.

Joel Anderson had a series of papers relating it to a conjecture in matrix theory. He could show that instead of a Hilbert space of infinite sequences, the problem could be stated in an equivalent form with finite dimensional Hilbert spaces, the so-called paving conjecture: For each $\epsilon > 0$ there is a number $r \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, an $n \times n$ matrix with vanishing diagonal can be partitioned into at most r sets and when the matrix is restricted to the rows and columns from each set in the partition, then the norm of the resulting principal submatrix is at most $1 - \epsilon$ times the norm of the matrix. Here, the norm of an $m \times m$ matrix is the operator norm when the matrix acts on the Euclidean space $\ell^2(\{1, 2, \ldots, m\})$.

The proof by Marcus, Spielman and Srivastava [MSS] builds on the paving conjecture and recent work by Nik Weaver [Weaver]. Motivated by discrepancy theory, Weaver proved that Kadison-Singer is equivalent to a statement about tight frames: For some $r \geq 2$, there are universal constants $\eta \geq 2$ and $\theta > 0$ such that every finite tight frame, with frame vectors whose norms are bounded by one and with frame bound η, can be partitioned into r sets, each of which have upper frame bound $\eta - \theta$. One intriguing aspect of this formulation is that, while it was known to be impossible to pave matrices using partitions of size two, Weaver's version was open even for $r = 2$. Indeed, Marcus, Spielman and Srivastava prove Weaver's formulation with constants $\eta = 18$, $\theta = 2$ and $r = 2$.

Another constituent of the proof is the literature on the Polya-Schur-Lax Problems [LPR], Hyperbolicity and Stability Preservers by the late Julius Borcea and by Petter Brändén [BB] (and the topic of a previous AIM workshop). Motivated by these results, Marcus, Spielman and Srivastava prove that the characteristic polynomials that are relevant for Weaver's conjecture form an interlacing family (a concept introduced by them, which is closely related to interlacing polynomials, i.e. real rooted polynomials whose roots alternate) by showing that these polynomials can be realized as a stability preserving transformation of a real stable polynomial. Marcus, Spielman and Srivastava control the largest root (eigenvalue) of the expected characteristic polynomial using a multivariate barrier argument. As stated by the authors, the idea is to have an upper bound on the roots of the characteristic polynomial along with a measure of how far above the roots the upper bound is. Then, the authors are able to retain control of this bound after applying differential operators which give the form of the expected characteristic polynomial.

One goal of the workshop was to broaden the recent proof of the Kadison-Singer Problem and to explore its consequences. With the Kadison-Singer Problem having equivalent formulations in a dozen areas of mathematics, it became important to bring some of these diverse groups together to explore the consequences of the solution to all these areas of mathematics. Also, since the final proof came from an area far from where the problem originated, it became imperative that the *solvers* of the problem get together with the *receivers of the solution*. A strong interaction between researchers in the areas connected by this problem and by its proof should enlarge our understanding and to address further open problems.

A number of research groups were formed to work on some of the open problems surrounding the solution of the Kadison-Singer Problem. One such group included Bownik, Casazza, and Speegle who decided to try to find the best available solution to the Feichtinger Conjecture derivable from the work of Marcus/Spielman/Srivastava. They made some very good progress on this. In the meantime, Marcus was working to improve the solution which rests on showing that all unit norm η-tight frames (for $\eta = 18$) can be partitioned into two frames which have *good frame bounds*. The question is: how small can η be? It is know that $\eta = 2$ fails this result. Marcus eventually showed that $\eta = 4 + \epsilon$ works and Bownik, Casazza, Marcus, and Speegle eventually joined forces to combine their results into a paper - which is in progress.

Motivated by a talk by Anna Gilbert, another group decided to investigate the possibility of using the MSS technique for the construction of deterministic matrices with the restricted isometry property for sparse recovery. It was decided that perhaps the most promising direction would be the case of frequency measurements, which might be amenable to an inductive selection principle that could possible improve on the randomized selection. Nevertheless, here was a general agreement that the control of spectral properties of a family of submatrices was qualitatively different from selecting just one submatrix.

Another group led by Bill Johnson was interested in whether knowing paving for ℓ_2 would give the bounds for paving in ℓ_p. Eventually, they concluded it does not. This group also found what appeared to be a totally new proof for computing the distance from a Banach space to the cube. After scouring the literature, the decided that this proof could be gleened from existing works (with some effort).

A group that wanted to understand how to translate the methods developed in [MSS] into other areas (particularly analysis) was formed, and significant progress was made. Much of the polynomial machinery from [MSS] was translated into shifts of rational functions, giving a basis for understanding the proof in analytic terms. One important conclusions came from this: the bottleneck in turning the proof completely into analysis occurs at the point where one would need quantitative bounds on the resolvant. This could be the most impactful discovery of the workshop, because it now gives a clear understanding of how the methods in [MSS] could be used in a variety of other problems. Essentially, analysts can use the resolvant as a way of trying to understand the behavior of operators and, if they could do so, could then appeal to the polynomial machinery in [MSS] to get bounds that were previously unobtainable.

Another group formed in an attempt to understand whether an algorithmic version of the methods in [MSS] could be made. They showed that the original proof (which used total independence of the random vectors to form the mixed characteristic polynomial) only needed d-wise independence. This led to the first known algorithm that is polynomial in d (the dimension). This may be particularly important due to a recent result on the asymmetric traveling salesman problem in [AGTSP] that used an extension of the methods from [MSS] from the paper [AGKS]. This puts theoretical computer science in a somewhat interesting position, as is this the first notable case in no known approximation algorithm for a problem is as good as the LP relaxation. The likely outcome of this is that new algorithms will be found that DO match the LP relaxation bounds, and the ideas developed by this group will likely appear in any such algorithm.

On the last day of the workshop Leonid Gurvits pointed out connections between the use of hyperbolic polynomials in Van der Waerden-like conjectures and topics in statistical mechanics by Elliott Lieb and Alan Sokal. He also discussed some ideas on how to develop a general theory in this area — one that would encompass both [MSS] and his proof of the Van der Wearden conjecture. He posed some problems that would cast some light in that direction.

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