Automorphic kernel functions

organized by
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Workshop Summary

The AIM workshop on Automorphic Kernel Functions was a fantastic success. It brought together world experts from three different continents dedicated to the study of automorphic representations into an environment ideal for collaboration on problems that should shape the future of the field. The workshop focused on the efforts of a variety of seminal mathematicians to elucidate the structure of endoscopic transfers of automorphic representations and to move beyond the known theory of twisted endoscopic transfers.

The Langlands functoriality conjecture represents a profound unifying force in modern mathematics. Over the last forty years, progress has been made mainly towards the theory of *Endoscopy* for classical groups, plus lower rank, but very striking cases of functoriality, which can now be thought of as cases of *Beyond Endoscopy*. The trace formula is the primary tool used to establish the endoscopic classification of the automorphic discrete spectrum for classical groups. This establishes Langlands functoriality from classical groups to general linear groups with respect to very special choices of morphisms of L-groups. When the cuspidal automorphic representations on the quasisplit classical groups have non-zero Whittaker models, these functorial transfers can also be proved by the Converse Theorem and L-function method, which is closely related to the Langlands-Shahidi method for L-functions, and also by an explicit construction method, called automorphic descent, which is closely related to the Rankin-Selberg method for L-functions. Langlands' *Beyond Endoscopy Proposal* is, in some sense, to combine the three approaches together in order to establish functoriality in general, which implies, for instance, the generalized Ramanujan conjecture.

The activities in the workshop followed the spirit of the Langlands' *Beyond Endoscopy Proposal*. The talks on Monday were to report on the state of art of the theory of endoscopy via the trace formula approach by Tasho Kaletha, and on the explicit construction of endoscopic transfers via integral transforms with automorphic kernel functions by Baiying Liu. In the remaining four days, the workshop focused on how to move from existing results to Beyond Endoscopy.

One of the key notions discussed in the workshop is the notion of a Basic Functions in the sense of Braverman-Kazhdan and Ngô. This theory aims to generalize the usual Fourier transform on the space of $n \times n$ matrices, used by Godement and Jacquet to prove the functional equation of the standard L-function of an automorphic representation, to a setting appropriate for studying the Langlands' L-function $L(s, \pi, \rho)$. One has a definition of the basic function for an arbitrary ρ, which in the case of the $n \times n$ matrices is the characteristic function of $\mathfrak{gl}_n(\mathcal{O}_F)$ for $\mathcal{O}_F$ the ring of integers of a nonarchimedian local field F. The two primary local problems at this point are giving a good interpretation of this basic function and finding a space of ρ-Schwartz functions in which it lives. This involves the introduction of a monoid attached to the ambient group that generalizes the embedding

$GL_n \hookrightarrow \mathfrak{gl}_n$ of semigroups. The relevant lectures by Lei Zhang and by Freydoon Shahidi were very educational. The discussion group on basic functions or more generally on the ρ-Schwartz functions focused on asymptotic behavior of the ρ-Schwartz functions on monoids, the ρ-Fourier transform (which should generalize the usual Fourier transform on $\mathfrak{gl}_n$, viewed as a vector space) and their relations to the Langlands-Shahidi method and the Rankin-Selberg method. Dihua Jiang, Freydoon Shahidi, and Lei Zhang will continue to work on some basic cases along these lines.

Lectures by Altug and Matz explained the difficulties involved in studying nonstandard test functions via the trace formula following Langlands' suggested approach to Beyond Endoscopy. Motivated by this and the complementary lecture of Sakellaridis, the discussion group led by Sakellaridis began systematically writing down cases in which the beyond endoscopy proposal has been successfully executed. The idea was to try and phrase the arguments in a way that makes them uniform, and as close as possible to the usual paradigm of local global comparisons that are traditional components of trace formula comparisons. Sakellaridis' discussion group will certainly continue their work beyond the workshop.

The lectures on relative trace formula in the spirit of Beyond Endoscopy brought fresh air to the subject. They explained how the nature of functoriality Beyond Endoscopy is closely related to classical problems, such as estimate of sums of Kloosterman sums or integrals. In a letter of Sarnak to Langlands, Sarnak explained a useful analytic benchmark that explains precisely why beyond endoscopy is harder than endoscopy from the point of view of analytic number theory. Using this benchmark as a guide, Getz explained a possible way to produce a relative trace formula that should allow one to go beyond the cases of functoriality that have been obtained using the usual trace formula or Kuznetsov formula. The relevant discussion group was focused on understanding the spectral meaning of such a relative trace formula. It appears that it involves an intertwining operator between automorphic forms on nonreductive groups.

The lecture by Ngô provided his overview of Langlands' Beyond Endoscopy Proposal. He explained what has been done and feasible problems for the near future. He highlighted establishing formulae for the kernel functions occuring in local non-endoscopic transforms. In the $SL(2)$ case these formulae have been proved by Langlands and stated by Gelfand, Graev and Piatetski-Shapiro, and Ngô's student Johnstone has provided generalizations in higher rank. A discussion group considered some of his suggestions and gave outlines of proofs in some cases.

Finally, the organizers invited four graduate student participants to present 20-minute lecture on their thesis work. The lectures were very exciting. The workshop was an ideal opportunity for them to see the past and the future of the subject. We are very happy to have such strong young experts joining the force dedicated to realizing Langlands' Beyond Endoscopy proposal.