Automorphic forms and harmonic analysis on covering groups

organized by
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Workshop Summary

In recent years, there has been much progress in the Langlands' program i.e. theory of automorphic forms and representations of reductive algebraic groups over local fields. A key ingredient in the theory is the Langlands' dual group. Roughly speaking, the dual group tells us how to organize representations into packets and how to transfer them from one group to another (the functoriality principle). In fact, a huge amount of effort was needed just to formulate precise conjectures.

Reductive algebraic groups over local and global fields have topological central extensions. A well known example is the metaplectic group, i.e. the 2-fold central extension of the symplectic group. Although these groups are not linear, they are playing an important role in the theory of automorphic forms and representations. For example, classical theta series of half-integral weight are examples of automorphic forms on the metaplectic group. However, the covering groups are technically not a part of the Langlands' program. There is no notion of the dual group and, it is safe to say, we still do not understand the statements that are expected to hold. The aim of this workshop was to overview recent developments in the subject and to make a progress towards the general picture.

The workshop was organized in the usual manner, with lectures in the morning, and research groups in the afternoon. The first lecture was given by Gopal Prasad. Its aim was to give a survey of central extension of reductive groups. The subsequent lectures gave an overview of the relevant, known, results. Jeff Adams and Peter Trapa gave talks on real groups. Many of the usual methods in the representation of real groups are applicable to non-linear groups. Thus, the overall picture is very explicit, and quite advanced when compared to p-adic groups. Wen Wei Li and Tamotsu Ikeda gave talks on the stabilization of the trace formula for the metaplectic group and the n-fold central extension of SL_2, respectively. Wee Teck Gan gave a survey of results in the theta correspondence for the metaplectic group. Marty Weissman gave a talk on the conjectural definition of the Langlands' dual group. One possible justification for the definition of the dual group comes from Hecke algebra isomorphisms, and was the topic of Gordan Savin's lecture. Friday lectures were given by two recent PhD's: Wan-Yu Tsai and Daniel Szpruch.

Some, more notable, topics discussed by research groups were:

Brilynski-Deligne central extensions and automorphic forms on metaplectic tori..
It is well known that only perfect groups admit universal central extensions. Simply connected, semi-simple groups are perfect (over most fields), and the corresponding universal central extensions have been well understood. For general reductive groups, Brylinski and Deligne have introduced a class of central extensions, that includes the universal central extension, if the group is simply connected. This class of central extensions seems to be

a natural setting for the theory of automorphic forms and representations. A group lead by Marty Weissman and Tamotsu Ikeda studied automorphic forms on a Brylinski-Delgine cover of a (non-split) torus. The classification of (genuine) automorphic forms reduced to finding a maximal abelian subgroup. A nice answer to this problem has been found.

Hecke algebras and Plancherel measures.
One justification for the definition of the Langlands' dual group is via the isomorphisms of Hecke algebras. For example, the Bernstein's component for the metaplectic group, containing the even Weil representation, is equivalent, via an explicit Hecke algebra isomorphism, to the Bernstein's component for the split $SO(2n+1)$, containing the trivial representation. This component, for $SO(2n+1)$, contains a remarkable square integrable representation, called the Steinberg representation. Thus, via the equivalence of categories, we define a Steinberg representation of the metaplectic group. An obvious, but important question, is to compute the formal degree of the Steinberg representation of the metaplectic group. This problem was considered by the group lead by Gordan Savin. An important remark was made by Trapa, and a more general approach emerged: if the Hecke algebra preserves the natural L^2-norm, then the algebra isomorphism transfers the Plancherel measure from one group to another. This is certainly true for the metaplectic group. In general, when the isomorphism of Hecke algebras is given in terms of Bernstein's generators, the invariance of the L^2-norm is not obvious, but this leaves us with a good post-conference problem.

Transfer Factors and Endoscopy..

The theory of endoscopy plays a central role in the Langlands program. This enters when studying harmonic analysis on a group G: one can understand character formulas and orbital integrals on G in terms of simpler "stable" objects on certain smaller endoscopic groups. A key ingredient of this formalism is the theory of transfer factors. These are well understood, although very technical, for linear groups.

In extending the Langlands program to nonlinear groups one of the first obstacles is that the notion of "stable" is not generally well defined in this setting. Although analogues of stability, transfer and endoscopy do arise for nonlinear groups they are not well understood. One aspect of this program is the theory of lifting from a linear to a nonlinear group, which came up in a number of the talks.

The simplest case is that of the metaplectic group, which behaves in some ways very much like a linear group. A key role is played by the character of the *difference* of the two halves of the oscillator representation. In fact this virtual character is the transfer factor for the metaplectic group, and can be used to define lifting, stability, and endoscopy. See the section on Character Identities for $Mp(2n)$.

We had several extended working sessions dedicated to understanding transfer factors for nonlinear groups. This included some discussion of a new understanding of classical transfer factors for linear groups, using insight from the metaplectic group. Adams led these sessions, with important input from Tasho Kaletha and Wen Wei Li.

Character Identities for $Mp(2n)$..
While the theory of endoscopy is not known for general covering groups, it has been developed for $Mp(2n)$ in the thesis of Wen-Wei Li. In his talk at the workshop, Li described the stabilisation of the elliptic part of the trace formula for $Mp(2n)$. The fully stabilised trace

formula should lead to the definition of L-packets for $Mp(2n)$ defined via trace formula techniques. On the other hand, Gan and Savin have given a construction of L-packets for $Mp(2n)$ using theta correspondence from $SO(2n+1)$. It is thus a natural question whether these two constructions give the same thing; in other words, whether the L-packets of Gan-Savin satisfy the endoscopic character identities.

This problem was considered by a working group led by Gan on Thursday afternoon. The main work is to globalize a discrete series L-packet of $Mp(2n)$ so that the resulting global cuspidal representations have nonzero global theta lifts to $SO(2n+1)$. Another key step is to exploit the partially stabilised trace formula of $Mp(2n)$ to extract the local character identities. Many of these details were worked out during the discussions, but there were a few technical difficulties left, and several further inputs needed from the theory of endoscopy. Gan and Li are collaborating on a project to iron out these last difficulties.