Computing arithmetic spectra

organized by
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Workshop Summary

The purpose of this workshop was to extend current methods, and explore new methods, for computing Maass forms.

Maass forms are real analytic functions on the upper half-plane which are eigenfunctions of the (hyperbolic) Laplacian and are invariant under a cofinite subgroup of $SL(2,\mathbb{R})$. The motivation for studying these functions comes from number theory and the most interesting cases are the Hecke congruence groups

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \ : \ N|c \right\}. \tag{1}$$

The motivation for this workshop was recent work, by several groups of researchers, on the numerical computation of Maass forms and Laplacian eigenvalues for subgroups of $SL(2,\mathbb{Z})$. Some of those methods rigorously determine the Maass form, or at least the Laplacian eigenvalue. Some of the methods have only been applied to the case of the full modular group $\Gamma_0(1)$, while others seem to work for a wide variety of groups. Understanding these methods and how to extend them was the theme of the workshop.

Talks on the first day provided an introduction to Maass forms on GL(2) and an introduction to L-functions. Talks on the second day gave introductions to Maass forms on GL(3) and to the trace formula for GL(2).

Starting on Wednesday there were more specialized talks on using the converse theorem to locate GL(3) Maass forms, the numerical computation of L-functions, using L-functions to find GL(3) Maass forms, and the Voronoi summation formula.

Afternoons were devoted to discussions and group work. The remainder of this report is a description of some of the discussions.

1. Rigorous determination of Maass forms

Work of Booker, Strömbergsson, and Venkatesh has shown that, for the first few cases, the Maass forms which have been numerically discovered for the full modular group are indeed correct. The approach requires the Laplacian eigenvalue and the first several Fourier coefficients to high precision. The method makes use of the shape of the fundamental domain of $SL(2,\mathbb{Z})$ and relies heavily on the Hecke operators.

Discussions concerned the extension of this work to $\Gamma_0(N)$. The consensus was that the case of prime N almost certainly was tractable because it is easy to describe a nice fundamental domain. And probably the case of squarefree N also is tractable because the completed group ($\Gamma_0(N)$ along with all the Fricke/Atkin-Lehner involutions) has only one cusp. Fredrik Strömberg didn't think non-squarefree level would be that much more difficult if one had some experience working with Fourier expansions at several cusps.

2. Certification of class numbers

A collaboration between Booker, Michael Jacobson and Strömbergsson Andrew Shallue was started at the workshop with the aim of certifying long lists of class numbers of real quadratic fields. By known techniques, for each real quadratic field $\mathbf{Q}(\sqrt{d})$ one can compute in time $O(d^{\frac{1}{4}+\varepsilon})$ a number which under GRH is equal to the class number $h(d)$, and which unconditionally is a lower bound of $h(d)$. The aim of the present project is to give an algorithm to *certify* a long list of such numbers, using the Selberg trace formula for Hecke eigenvalues of Maass forms, and a positivity argument. The computations will depend crucially on several recent developments in effective computations of Maass forms, as it will require input of long lists of certified Maass form Hecke eigenvalues. On the theoretical side the algorithm is the best known today for unconditional computation, and in practice the hope is to be able to compute $h(d)$ for all $0 < d < 10^{11}$ and beyond.

3. Higher rank groups

Considerable discussion followed the talk by Ce Bian on his work, under the direction of Andy Booker, that uses the converse theorem to find Maass forms on $SL(3, \mathbb{Z})$. The approach involves solving an overdetermined linear system in several thousand unknowns. AIM prepared a press release about this work, which was picked up by several news outlets, some of which mistakenly reported that these calculations would help prove the Riemann Hypothesis! (This was probably due to a misinterpretation of the fact that Mike Rubinstein checked the Riemann Hypothesis for the first several zeros of the L-function associated to one of Bian's Maass forms.)

Stefan Lemurell gave a talk about his work with Farmer and Koutsoliotas that uses one functional equation, and the Euler product, to find $SL(3, \mathbb{Z})$ Maass forms. Their method has the feature of only requiring a small system of equations, but it is not clear how far the calculations can be pushed. It was during the workshop, with input from discussions and the success of Booker and Bian, that they were able to get the method to work.

Discussions focused on possible extensions and other methods that could be used for GL(3) Maass forms. Boris Mezhericher, a finishing graduate student of Dorian Goldfeld, described his approach that involves multiplying the L-function by a hypergeometric series before doing an inverse Mellin transform. This leads to a function that satisfies a GL(2)-type transformation law, so it is possible that methods similar to those used by Hejhal can be applied.

Goldfeld suggested dividing the GL(3) L-function by (a translate of) the Riemann zeta function to obtain a degree 2 functional equation. The cost is that the values of the L-function at the zeros of the zeta-function are now unknowns, so it is not clear if this will lead to a simpler set of equations.

It also was noted that nobody had tried directly working with the Fourier expansion of the GL(3) Maass form. Various difficulties, such as evaluating the Whittaker function and the embedded GL(2) sum, were discussed. Mezhericher indicated that he had computer code for the Whittaker function, and Strömberg said that he might consider that approach "if nobody else does it."

Dan Bump suggested that after GL(3) the next easiest case is Sp(4).

4. Trace formula methods

Booker, Strömberg and Strömbergsson have used the trace formula for Hecke operators to rigorously determine the first several eigenvalues for $\Gamma_0(N)$ with a character $\chi \mod N$, for small N. That work is ongoing. In a related vein there is the problem of generalizing the numerical verification carried out by Booker and Strömbergsson (using the trace formula, but not involving Hecke operators) of the Selberg eigenvalue conjecture to squarefull N. Strömbergsson is currently working also on this project.

5. Converse theorems

A small group discussed methods for proving converse theorems without twists, in the hopes that such a result could be proven for $SL(3, \mathbb{Z})$. It did not appear that existing methods could be extended beyond the few known cases.

6. Infinite volume

Very few, if any, computations have been done in a case where the quotient space has infinite volume. A specific problem was suggested by Elena Fuchs. The group in question is related to the Apollonian sphere packing and gives rise to an infinite volume quotient in hyperbolic three space. The primary question from Elena is to determine the (non-)existence of a second small eigenvalue. During the week there were discussions on ways to attack the problem theoretically and Holger Then started to try to adapt his program for the hyperbolic three space to handle this case.