Arborealization of singularities of Lagrangian skeleta

organized by
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Workshop Summary

1. Overview

The workshop was devoted to the study of Weinstein manifolds, the symplectic counterparts of affine complex (Stein) manifolds [CE]. Any Weinstein manifold M deformation retracts to a singular Lagrangian complex L called its skeleton. When the skeleton is smooth, the Weinstein manifold is its cotangent bundle $M \simeq T^*L$. In this sense, a general Weinstein manifold can be viewed as the "cotangent bundle" of its singular skeleton.

The workshop focused on questions related to the theme: *Can the symplectic geometry of a Weinstein manifold M be recovered from the intrinsic smooth topology of its skeleton L?* In general the skeleton may have complicated singularities, but work in progress [AENS], presented at the workshop, shows that, up to Weinstein homotopy, one can arrange for the skeleton to have arboreal singularities [Narb]. Conversely, any complex with arboreal singularities uniquely determines a surrounding Weinstein manifold for which it is the skeleton. A primary goal of the workshop was to develop and apply the notion of arboreal complexes as a natural generalization of smooth manifolds.

Participants came with expertise in diverse subjects – symplectic and contact geometry, mirror symmetry, sheaf theory, and singularity theory, to name a few – and the workshop aimed to integrate their viewpoints. There were lively discussions about the potential of different topological tools – such as arboreal skeleta, microlocal sheaves, and tête-à-tête complexes – to impact longstanding and more recent problems in symplectic geometry – notably, Reeb dynamics of Liouville sectors/Weinstein hypersurfaces/Reeb stops, geometry of fronts, Fukaya categories, wall-crossing, and flexibility.

The workshop followed the AIM model of formal talks in the morning, followed by problem sessions in the afternoon. Specific directions of emphasis, as expanded on below, included:

(1) Symplectic invariants via arboreal skeleta.
(2) Combinatorial mutations and symplectomorphisms.
(3) Reeb dynamics near arboreal skeleta.
(4) Applications of microlocal sheaves to fronts.

During Monday afternoon, Kai Cieliebak led a discussion of open problems, with notes by Ben Gammage, and participants settled on directions to tackle during the rest of the week.

2. Morning talks

The morning talks fit roughly into the following themes.

2.1 Arboreal skeleta There were introductory talks by Nadler and Starkston explaining the basics of arboreal singularities and skeleta. Participants were particularly interested in their symmetries and elaborations adapted to different contexts. Eliashberg gave a talk summarizing the status of the work in progress [AENS] that one can arrange for any Weinstein manifold to have an arboreal skeleton.

2.2 Symplectic invariants Pardon gave a talk summarizing the status of his joint work in progress with Ganatra and Shende [GPS1, GPS2] localizing the (wrapped) Fukaya category to a skeleton. He discussed how in the case of an arboreal skeleton one can calculate the resulting (co)sheaf. Shende gave a talk introducing the alternative language of microlocal sheaves and its relation to Fukaya categories. Abouzaid gave a talk advertising related challenges found within mirror symmetry, specifically wall-crossing formulas.

2.3 Mutations and symplectomorphisms A'Campo gave an inspiring talk about Lefschetz curve fibrations, focusing on the combinatorial construction of monodromies via Thom-Sebastiani factorization. He put the theory in the more general context of tête-à-tête complexes [A]. Portilla Cuadrado further elaborated on open problems and more detailed constructions in this direction. Zorn gave a talk about the classification of combinatorial mutations of arboreal skeleta found in his thesis [zorn].

2.4 Flexibility Murphy gave a talk reporting on her ongoing work to characterize flexibility in terms of microlocal sheaves.

3. Afternoon Problem Sessions

The afternoon problem sessions pursued the following questions.

3.1 Arboreal skeleta and invariants in low-dimensions

The theory of skeleta of Weinstein manifolds in dimension two (ribbon graphs) and dimension four (Stokes data, cluster transformations) is extraordinarily rich. One group pursued specific calculations in this context, for example microlocal sheaves along arboreal skeleta with vanishing top homology such as the contractible Bing's room. They also took on the challenge of finding explicit arboreal skeleta starting from three and four-dimensional singular thimbles.

3.2 Reeb dynamics near arboreal skeleta The relative or local theory of Weinstein manifolds can be formulated alternatively in terms of Liouville sectors, Weinstein hypersurfaces, or Reeb stops. Questions typically depend upon the Reeb dynamics around a singular Lagrangian, or the effect on the Reeb dynamics of the insertion of a singular Legendrian. One group focused on the Reeb dynamics near arboreal singularities, with connections to their appearance in Lefschetz fibrations.

3.3 Tête-à-tête symplectomorphisms Motivated by the monodromy of vanishing cycles in Lefschetz fibrations, A'Campo introduced a notion of tête-à-tête symplectomorphisms. They are combinatorial creations associated to certain simplicial complexes, generalizing Dehn twists around Lagrangian spheres. One group sought to expand the scope of tête-à-tête symplectomorphisms to new classes of skeleta, in particular those with simplicial or arboreal singularities.

3.4 Geometry of manifolds and fronts It is a longstanding mystery whether the cotangent bundles of smooth manifolds, viewed as Weinstein manifolds, distinguish different smooth structures. This can be reformulated in terms of the fronts of Legendrians, in particular whether isotopic Legendrians produce diffeomorphic fronts. One group pursued the structure of such fronts with the aim of their smooth parameterization. There was particular enthusiasm that the barcodes (à la persistent homology) provided by microlocal sheaves offer new, untapped structure.

3.5 Combinatorics of arboreal singularities There are beautiful symmetries and mutations of arboreal singularities and their Weinstein neighborhoods. At the level of invariants, these connect with quiver representations and their reflection functors. In his thesis, Zorn mapped out the symmetries and mutations of arboreal singularities, providing a higher-dimensional theory of ribbon graphs and their moves. One group discussed questions resolved and raised by this work, in particular how to lift the combinatorics back to geometry.

Overall the workshop was quite productive in starting new directions of research and new collaborations.

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