Implementing algebraic geometry algorithms

organized by
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Workshop Summary

Introduction. *Macaulay 2*, developed by Dan Grayson and Mike Stillman, is one of the leading computer algebra programs for research and teaching in algebraic geometry and commutative algebra. *Macaulay 2* has a structure called a package, which is designed to encourage, ease and expand contributions to *Macaulay 2*.

The main goal of this workshop was to write *Macaulay 2* packages implementing algorithms in the following rapidly growing areas:

- algebraic statistics (AS);
- numerical algebraic geometry (NAG);
- toric geometry (TG).

Here "writing a *Macaulay 2* package" means not only implementing related algorithms but also writing the corresponding documentation and tests for the code.

The above-mentioned areas have significant applications in fields as diverse as computational biology, coding theory, statistics and string theory. We hope, therefore, that the packages developed through this workshop will advance our knowledge and understanding across multiple disciplines. In the following paragraphs, we summarize the activities and achievements during the workshop.

Activities. On the first day, after presentations designed to develop a list of goals, we broke into three groups AS, NAG and TG. Due to the considerable overlap between these three area, there were interactions between the groups, but the members of each group generally stayed fixed during the workshop week. We started each morning session with a progress report from each group with members from other groups often asking questions and making suggestions. After the progress reports participants worked in groups on packages, consulting with other members of the workshop as needed. At the beginning of the afternoon session, we discussed various *Macaulay 2* related issues including "How-To" sessions. The items discussed during that period of time were recorded and posted at the workshop wiki. The wiki page can be found at the following URL:

`http://wiki.macaulay2.com/AIM2009/`

The remaining afternoon session was devoted writing packages with workshop members often working quite late. Below we highlight important events that occurred during the workshop.

As we mentioned, the morning session of the first day of the workshop was aimed at formulating a list of problems in each of the three areas of focus. There were introductory talks by Brandilyn Stigler and Sonja Petrovic (AS), Diane Maclagan (TG) and Anton Leykin

(NAG). The presentations identified *Macaulay 2* packages which exist, but need improvement and the speakers suggested the creation of the following packages:

Algebraic Statistics

(1) studying connectivity of fibers of a map associated to varieties in statistical applications with the gaol of extending our ability to study such varieties;
(2) implementing Luis Garcia's `Singular` library "Markov.lib" in *Macaulay 2* and extend it to more graph types and for Gaussian models;
(3) identifiability of parameters in an algebraic statistical model;
(4) computing Markov subbases;

Numerical Algebraic Geometry

(1) solving zero-dimensional polynomial systems;
(2) tracking user-defined homotopies;
(3) positive-dimensional solution sets: irreducible and primary decompositions;
(4) Schubert calculus applications such as the Littlewood-Richardson homotopy and the Pieri homotopy;
(5) recovering exact data from numerical approximations and certification;
(6) interfacing with other numerical polynomial homotopy continuation software;

Toric Geometry

(1) toric sheaf cohomology;
(2) intersection theory;
(3) resolution of singularities;
(3) Nash blow-up;
(4) rings of piecewise polynomials;
(5) maps between toric varieties.

In the afternoon session of the fourth day of the workshop, each group reported their progress and demonstrated the packages resulted through the workshop. The AS group presented the new package `ToricFibers` and commented on the progress of two other new packages, `Graphs` and `GraphicalModels`. The NAG group displayed their new package `IdealOfGenericPoints` package and updated the group on progress made in the package `NumericalAlgebraicGeometry`. Finally, the TG presented their new package `ToricResolve` designed to perform resolution of singularities of toric varieties.

At the final morning session, each participant proposed their short-term and long-term goals related to the projects during this workshop (an activity found to be useful by the organizers at other workshops). Some of the projects initiated at this workshop will be discussed and explored further at a *Macaulay 2* workshop at MSRI, January 8–12, 2010.

Achievements

Algebraic Statistics

- Given a design (dxn)-matrix A of a normal toric model the new package `ToricFiber` provides functions for building and working with a graph describing the fiber for all possible vectors b in the cone(A) such that b is a linear combination of at most d columns of A. Such fibers play a key role in one possible algorithm for computing

minimal generators of a toric ideal and may lead to a more efficient way of studying the model rather than computing the full minimal generating set. The package allows users to (a) study if there is yet a more efficient algorithm for computing these toric fibers and (b) if there is a special structure in the case of statistical models that allows even more efficient construction of the information required to study the models.

- A rough package for working with discrete models described by conditional independence statements represented by a directed acyclic graph `Markov` was implemented by Luis Garcia in both *Macaulay 2* and *Singular* — which often given different answers to problems. The group determined why different answers occur, developed a new package `Graphs` to make uniform the use of graphs by several packages, including this one, and began writing code to include undirected graphs and Gaussian models. Finally, the package was renamed `GraphicalModels`.

Numerical Algebraic Geometry

- Numerical Schubert Calculus: Numerical Schubert Calculus is used to numerically solve problems in enumerative geometry. The NAG group worked on the `NumericalSchubertCalculus` package that uses the Littlewood-Richardson homotopy and the Pieri homotopy.

  The Littlewood-Richardson (LR) homotopy in PHCpack was recently released as version 2.3.52 (PHCpack is a general-purpose solver for polynomial systems by homotopy continuation). The *Macaulay 2* package `LRhomotopies`, which provides an interface to the LR homotopy in PHCpack, was completed.

  The function `solveSimpleSchubert` uses the Pieri homotopy to find all solutions to simple Schubert problems. It uses algorithms described in the paper "Galois groups of Schubert problems via homotopy continuation", Mathematics of Computation, 78 (2009) 1749–1765, by Anton Leykin and Frank Sottile.
- Interfacing with PHCpack: In addition to LRhomotopies.m2, the multiprecision root refiners of PHCpack were also improved during the workshop and used to compute accurate generic points via the `NumericalAlgebraicGeometry` package in *Macaulay 2*.
- Recovery of exact data from numerical approximations: One of the advantages of numerical algebraic geometry is that it is simple to produce approximations of generic points on any given algebraic variety. The NAG group worked on a package called `IdealOfGenericPoints` which recovers exact defining equations for the variety from the approximated points.
- The NGA group started to build the foundation for dealing with the positive-dimensional case. In particular, the `WitnessSet` type has been contructed.
- The package `NumericalAlgebraicGeometry` was recently submitted to the Journal of Software for Algebra and Geometry (`http://j-sag.org`) for publication.

Toric Geometry

- The TG group worked on code for the `NormalToricVarieties` package, including developing intersection theory, implementing resolution of singularities (`ToricResolve`), and interfacing with the `Polyhedra` package. After its initial implementation, they were able to speed up the resolution of singularities by a factor of ten and discussed possibilities for future improvement.

- Some members of that group also worked on the package `ToricVectorBundles` which was recently submitted to the Journal of Software for Algebra and Geometry (`http://j-sag.org`) for publication.

Closing comment. Future work on packages will be recorded in a series of wiki pages that were planned at the AIM workshop. Those interested should watch the *Macaulay 2* page

`http://www.math.uiuc.edu/Macaulay2/`

and the *Macaulay 2* google group.