Holowinsky and Soundararajan's proof of the holomorphic QUE

Briefly, Holowinsky and Soundararajan prove that the values of holomorphic cusp forms of weight k with multiplicative coefficients become equidistributed over their fundamental domain as $k \to \infty$. The assumption that the forms have multiplicative coefficients is important, because the example $\Delta(z)^k$ is a weight $12k$ form for which scarring occurs as $k \to \infty$. It is also essential that the form be a cusp form. For example, the Eisenstein series $E_k(z)$ is a modular form of weight k with multiplicative coefficients, but it's value distribution also shows scarring (for example, it has all $k/12 + O(1)$ zeros situated along the arc of the unit circle in the fundamental domain.

Here are some more details. Let $f_k(z)$ be a holomorphic Hecke eigenform of weight k for the full modular group. In particular,

$$f_k(z) = \sum_{n=1}^{\infty} \lambda(n) n^{\frac{k-1}{2}} e(nz)$$

for $\Im z > 0$ where, by Deligne's theorem, $|\lambda(n)| \le \tau(n)$ satisfies the Ramanujan bound. Let $F_k(z) = y^k |f(z)|^2$ so that

$$F_k\left(\frac{az+b}{cz+d}\right) = F_k(z)$$

for all $a, b, c, d \in \mathbf{Z}$ with $ad - bc = 1$, i.e. F_k is invariant under the action of $SL(2, \mathbf{Z})$ on the upper-half-plane $\mathcal{H}$. Thus, we may regard F_k as a function which lives on the fundamental domain $\mathcal{F} = SL(2, \mathbf{Z}) \backslash \mathcal{H} = \{z = x + iy : -1/2 \le x \le 1/2 \text{ and } x^2 + y^2 \ge 1\}$.

Let $g(z)$ be a smooth bounded function on $\mathcal{F} =$ Then, Holowinsky and Soundararajan show that

$$\int_{\mathcal{F}} F_k(z) g(z) d\mu = \int_{\mathcal{F}} g(z) d\mu + o(1)$$

as $k \to \infty$. Here $d\mu = y^{-2} dx dy$ is the measure on the upper half-plane which is invariant under $SL(2, \mathbf{R})$. The interpretation is that the profile (or mass) of F_k becomes uniformly distributed on $\mathcal{F}$. This theorem is the holomorphic analog of the quantum unique ergodicity (QUE) conjecture of Rudnick and Sarnak.

As a beautiful corollary it follows by an argument of Rudnick that the zeros of $f_k(z)$, of which there are $\frac{k}{12} + O(1)$, are uniformly distributed in $\mathcal{F}$.

By the harmonic analysis of $\mathcal{F}$, to prove the theorem of Holowinsky and Soundararajan, it suffices to consider the cases where $g(z) = \phi(z)$ a Maass form or where $g(z) = E(1/2 + it, z)$ is the Eisenstein series. Here we will discuss the case that $g(z) = \phi(z)$ is a Maass form. Then

$$\phi(z) = \sum_{n=1}^{\infty} \rho(n) \sqrt{y} K_{ir}(2\pi n y) \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (2\pi n x),$$

for some r, where K is a Bessel function. In this case there is no main term and the object is to show that the integral $\int_{\mathcal{F}} \phi(z) F_k(z) d\mu$ tends to 0 as $k \to \infty$.

There are two approaches to calculating this integral. One is by way of Poincaré series, which leads to the study of the shifted convolution sums:

$$\sum_{n\leq x}\lambda(n)\lambda(n+\ell);$$

this is the approach that Holowinsky took, introducing sieve methods into the study of such sums, and exploiting the fact that the coefficients $|\lambda(n)|$ are small a lot of the time. The other is by Watson's formula:

$$\left(\int_{\mathcal{F}}\phi(z)F_k(z)d\mu\right)^2 = C_{k,\phi}\frac{L(1/2, f\times f\times\phi)}{L(\mathrm{sym}^2 f, 1)^2}$$

where $C_{k,\phi}\approx_\phi k^{-1}$; this is the starting point for Soundararajan. Here the L-function in the numerator is the triple product L-function of Garrett evaluated at the center of the critical strip and the L-function in the denominator is the symmetric square L-function associated with the cusp form f evaluated at the edge of the critical strip. Soundararajan found a new upper bound, which he calls a weak-subconvexity bound, for the numerator L-function.

The net result of Holowinsky's approach is that

$$\int_{\mathcal{F}}\phi(z)F_k(z)d\mu \ll \frac{\prod_{p\leq k}(1+|\lambda(p)|/p)}{(\log k)^{1-\epsilon}L(1,\mathrm{sym}^2 f)^{1/2}}.$$

Holowinsky proved that the right hand side tends to 0 with at most k^ϵ exceptions.

Soundararajan proved that $L(1/2, f\times f\times\phi)\ll\frac{k}{(\log k)^{1-\epsilon}}$ which nets

$$\int_{\mathcal{F}}\phi(z)F_k(z)d\mu \ll \frac{(\log k)^{\epsilon-1/2}}{L(1,\mathrm{sym}^2 f)}.$$

It is known that $\frac{1}{\log k}\ll L(1,\mathrm{sym}^2 f)\ll(\log k)^3$; the upper bound is standard, and the lower bound is due to Goldfeld, Hoffstein and Lieman and is tantamount to the assertion that there are no "Siegel zeros" for L-functions associated with cusp forms. Certainly, if $L(1,\mathrm{sym}^2 f)$ is not too small, then the right hand side of Soundararajan's estimate is small. Soundararajan can prove that this is the case for all but at most k^ϵ exceptions.

The question is: does the exceptional set of Holowinsky intersect the exceptional set of Soundararajan?

It turns out that the product over primes in the numerator of Holowinsky's bound can be estimated in terms of $L(1,\mathrm{sym}^2 f)$; the result is

$$\int_{\mathcal{F}}\phi(z)F_k(z)d\mu \ll (\log k)^{1/12+\epsilon}L(1,\mathrm{sym}^2 f)^{1/4}.$$

Thus, if $L(1,\mathrm{sym}^2 f)\ll(\log k)^{-1/3-\delta}$ for some $\delta>0$, then Holowinsky's bound does the job, whereas if $L(1,\mathrm{sym}^2 f)\gg(\log k)^{-1/2+\delta}$, then Soundararajan's bound does the job. Fortunately, there is plenty of room between 1/3 and 1/2, so such a δ exists and no cases are left out!