Qualitative Methods in Inverse Scattering

Fioralba Cakoni
cakoni@math.udel.edu

Department of Mathematical Sciences
University of Delaware, USA

Research supported by AFOSR
Approaches to inverse problems

- Weak scattering approximations:
  - multiple scattering is ignored, hence the problem is linear
  - a priori information is needed
Approaches to inverse problems

- **Weak scattering approximations**:  
  - multiple scattering is ignored, hence the problem is linear  
  - a priori information is needed

- **Optimization techniques**:  
  - multiple scattering is included, hence the problem is nonlinear  
  - a priori information and a good initial guess are needed  
  - only one or a few incident waves are needed and the reconstructions are reasonably good
Approaches to inverse problems

- Weak scattering approximations:
  - multiple scattering is ignored, hence the problem is linear
  - a priori information is needed

- Optimization techniques:
  - multiple scattering is included, hence the problem is nonlinear
  - a priori information and a good initial guess are needed
  - only one or a few incident waves are needed and the reconstructions are reasonably good

- Qualitative methods:
  - multiple scattering is included however the problem is linear
  - essentially no a priori information is needed
  - multi-static data is needed and only partial information about the scatterer is obtained
Obstacle Scattering Problem

Let \( \lambda \in L_{\infty}(\Gamma_I) \) and positive. The total field \( u \) satisfies

\[
\Delta_2 u + k^2 u = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \overline{D}
\]

\[
u u = 0 \quad \text{on} \quad \Gamma_D
\]

\[
\frac{\partial u}{\partial \nu} + i\lambda(x)u = 0 \quad \text{on} \quad \Gamma_I
\]

\[
u(x) = e^{ikx \cdot d} + u^s(x)
\]

\[
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial u^s}{\partial r} - ik u^s \right) = 0
\]
The scattered field $u^s$ has the asymptotic behaviour

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_{\infty}(\hat{x}, d) + O\left(r^{-3/2}\right)$$

as $r \to \infty$ where $r = |x|$, $\hat{x} = x/r$, $k$ is fixed and $u_{\infty}$ is the far field pattern of the scattered field $u^s$.

The inverse scattering problem is to determine the shape $D$ and the surface impedance $\lambda$, from a knowledge of $u_{\infty}(\hat{x}, d)$ for $\hat{x}, d \in \Omega$ and a fixed frequency $k$, where $\Omega$ is the unit circle.

In fact it suffices to know $u_{\infty}(\hat{x}, d)$ only for $d \in \Omega_1 \subset \Omega$ and $\hat{x} \in \Omega_2 \subset \Omega$. 
Uniqueness Theorems

Uniqueness of $D$

Theorem (Kirsch-Kress, 1993): $D$ is uniquely determined by $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in \Omega$ and a fixed value of the wave number $k$.

Theorem (Liu-Zou, 2007): If $D$ is a polygonal scatterer then $D$ is uniquely determined by $u_\infty(\hat{x}, d)$ for one $d$ and $\hat{x} \in \Omega$ and a fixed value of the wave number $k$.

Colton-Sleeman (1983), Gintides (2005): finitely many incident waves, Dirichlet boundary condition assuming a restriction on the size of the obstacle

Yamamoto, Alessandrini-Rondi, Yamamoto-Elschner: polygonal domains
Uniqueness Theorems

Uniqueness of $\lambda$

Theorem: $\lambda \in C(\overline{\Gamma_I})$ is uniquely determined from $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in \Omega$ and a fixed value of the wave number $k$.

The proof can be found in Cakoni-Colton book (2006)

Theorem (Colton-Cakoni-Monk, 2007): $\lambda \in L_\infty(\Gamma_I)$ is uniquely determined from $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in \Omega$ and a fixed value of the wave number $k$.

Remark: Since $u_\infty(\hat{x}, d)$ is an analytic function on $\hat{x}$ and $d$, in the above uniqueness results $\Omega$ can everywhere be replaced by a subset $\Omega_0 \subset \Omega$. 
Solution of the Inverse Problem

Determination of D

Newton’s Method Kress, Hohage, Potthast, Randell, Hettlich, Ganesh, Djellouli ....

We assume $u_\infty(\hat{x}, d)$ is known for $\hat{x} \in \Omega$ and one (or a few) $d$, and, for sake of simplicity, that $\partial D$ can be represented as

$$x = r(\hat{x})\hat{x}, \quad \hat{x} \in \Omega.$$  

Consider the mapping $F : r \rightarrow u_\infty$ and solve the non-linear and ill-posed equation

$$F(r) = u_\infty(\cdot, d).$$
Newton’s Method

Compute the Fréchet derivative $F'_r$ of $F : C^2(\Omega) \to L^2(\Omega)$.

- Kirsch, Potthast, Hettlich etc. for Dirichlet/ Neuman boundary condition.

The above equation is replaced by the linear equation

$$F(r) + F'_r q = u_\infty(\cdot, q)$$

which from an initial guess $r = r_0$ yields the new approximation $r_1 = r_0 + q$. Newton’s method consists in iterating this procedure.

Partial results on the convergence of the Newton Method are proven by Hohage (1998), Potthast (2001) ....
Qualitative Methods

or Non-iterative methods

- Find the shape, and extract information about the physical properties of the object.
- Rely on little a priori information and do it in a rather quick and simple way.
- Use multistatic data. Shape reconstruction is not very sharp and partial recovery of the physical properties is achieved.

A list of qualitative methods was presented by Potthast in his talk. In particular, the singular source method and range test for the reconstruction of $\mathcal{D}$ was described, see Potthast’s book (2001).

Our algorithm for solving the inverse scattering problem will be based on the linear sampling method (Colton-Kirsch 1996).
The Far Field Operator

Let \( \Phi(x, z) := \frac{i}{4} H_0^{(1)}(k|x - z|) \) which has the far field pattern

\[
\Phi_\infty(\hat{x}, z) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x} \cdot z}.
\]

Define the far field operator \( F : L^2(\Omega) \rightarrow L^2(\Omega) \) by

\[
(Fg)(\hat{x}) := \int_{\Omega} u_\infty(\hat{x}, d) g(d) \, ds(d).
\]

The Herglotz wave function with kernel \( g \) is defined by

\[
v_g(x) := \int_{\Omega} e^{ikx \cdot d} g(d) \, ds(d).
\]
The Far Field Equation

Define the far field equation

\[(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, z), \quad g \in L^2(\Omega), \quad z \in \mathbb{R}^2\]

Let \(z \in D\) and suppose that \(g\) solves the far field equation.

Rellich’s Lemma \(\implies u_g^s(x) = \Phi(x, z)\) in \(\mathbb{R}^2 \setminus \overline{D}\)

In particular \(-v_g = \Phi(x, z)\) on \(\Gamma_D\).

As \(z \in D \to \partial D\), \(\Phi(x, z) \to \infty\) and so does \(v_g \implies \|g\|_{L^2} \to \infty\).
Solving the Far Field Equation

Unfortunately, in general, the far field equation does not have a solution for any \( z \in \mathbb{R}^2 \! \! \!. \)

For \( z \in D \) the far field equation has a solution if and only if the interior boundary value problem

\[
\begin{align*}
\Delta v_z + k^2 v_z &= 0 & \text{in} & & D \\
v_z + \Phi(\cdot, z) &= 0 & \text{on} & & \Gamma_D \\
\frac{\partial(v_z + \Phi(\cdot, z))}{\partial \nu} + i\lambda(x)(v_z + \Phi(\cdot, z)) &= 0 & \text{on} & & \Gamma_I
\end{align*}
\]

has a solution \( v_z \) such that \( v_z = v_g \) is a Herglotz function with kernel \( g \).
Approximation by Herglotz function

Define

\[ \mathcal{H}(D) := \{ u \in H^1(D) : \Delta u + k^2 u = 0 \}. \]

**Theorem** (Colton-Sleeman (2001), Colton-Kress (2001)): Suppose that \( \mathbb{R}^2 \setminus \overline{D} \) is connected. Then the set of Herglotz functions

\[ \{ v_g : g \in L^2(\Omega) \} \]

is dense in \( \mathcal{H}(D) \).

**Theorem** (Cakoni-Colton (2001)): Suppose that \( \mathbb{R}^2 \setminus \overline{D} \) is connected. Then the set of Herglotz functions with kernel supported on a compact subset \( \Omega_0 \) of the unit sphere \( \Omega \)

\[ \{ v_g : g \in L^2(\Omega_0) \} \]

is dense in \( \mathcal{H}(D) \).
Solving the Far Field Equation

\[(Fg)(\hat{x}) = \Phi_{\infty}(\hat{x}, z)\]

For \(z \in D\) and a given \(\epsilon > 0\) there exists a \(g^\epsilon_z \in L^2(\Omega)\) such that

\[\|Fg^\epsilon_z - \Phi_{\infty}(\cdot, z)\|_{L^2(\Omega)} < \epsilon\]

and the Herglotz wave function \(v_{g^\epsilon_z}\) converges in \(H^1(D)\) to \(v_z\) where \(v_z\) is the solution of the interior mixed boundary value problem. Furthermore,

\[\lim_{z \to \partial D} \|v_{g^\epsilon_z}\|_{H^1(D)} = \infty \quad \text{and} \quad \lim_{z \to \partial D} \|g^\epsilon_z\|_{L^2(\Omega)} = \infty\]

For \(z \in \mathbb{R}^2 \setminus \overline{D}\) and a given \(\epsilon > 0\), every \(g^\epsilon_z \in L^2(\Omega)\) that satisfies

\[\|Fg^\epsilon_z - \Phi_{\infty}(\cdot, z)\|_{L^2(\Omega)} < \epsilon\]

is such that \(\lim_{\epsilon \to 0} \|v_{g^\epsilon_z}\|_{H^1(D)} = \infty\).
Indicator Function

Note that for $z \in D$, $v g^\varepsilon (z) \rightarrow v_z (z)$ point-wise.

- Construct a grid $\mathcal{G}$.
- For $z_i \in \mathcal{G}$, solve the regularized far field equation
  \[
  (\alpha I + F^* F) g_{z_i} = \Phi_\infty (\hat{x}, z)
  \]

The solution of the inverse problem is based on the use of the regularized solution $g_{z_i}$ of the far field equation and $v g_{z_i}$.

Open question: Does $g_{z_i}$ and $v g_{z_i}$ behave in the same way as the theoretical $g^\varepsilon_z$ and $v g^\varepsilon_z$?
Factorization Methods

By Kirsch (1998). The far field operator $F$ is replaced by $(FF^*)^{1/4}$. Then, if $F$ is normal (e.g. Dirichlet boundary condition)

$$\Phi_\infty(\cdot, z) \in \text{Range } (FF^*)^{1/4} \iff z \in D.$$  

Kirsch, Grinberg etc. has generalized these ideas to the case when $F$ is not normal, replacing $(FF^*)^{1/4}$ by the operator $F_\# := |\text{Re } F| + \text{Im } F$.

Arens and Lechleiter (2007) have proven when the far field operator is normal then

$$2\pi \| \varphi_z \|_{L^2(\Omega)}^2 < |vg_z(z)| < 4\pi \| \varphi_z \|_{L^2(\Omega)}^2$$

where $g_z$ is the regularized solution of $Fg_z = \Phi_\infty(\cdot, z)$ and $\varphi_z$ is the solution of $(FF^*)^{1/4} \varphi_z = \Phi_\infty(\cdot, z)$. 
Determination of $D$

To reconstruct $D$ we plot the level curves $\frac{1}{\|g_{zi}\|_{\ell^2}} = c$. The boundary is obtained for an appropriate choice of $c$.


$\|g\|$ with respect to $z$:
- Dirichlet boundary condition
- Impedance boundary condition
Determination of $\lambda$

This is based on the fact that $v_g$ approximates the solution $v_z$ of interior mixed boundary problem; Cakoni-Colton (2004)

- For every point $z \in D$ we have that

$$\int_{\Gamma_I} \lambda(x) |v_z(x) + \Phi(x, z)|^2 \, ds_x = -1/4 - \text{Im} (v_z(z)).$$

- Let $B_r \subset D$ be a ball of radius $r$ and denote by

$$\mathcal{V} := \{ f \in L^2(\partial D_I) : f = (v_z + \Phi(\cdot, z))|_{\Gamma_I}, \, z \in B_r \}.$$

Then $\mathcal{V}$ is complete in $L^2(\Gamma_I)$
Determination of $\lambda$

$$\Gamma_D = \text{Support}(v_z)$$

In particular if $\lambda$ is constant, we obtain

$$\lambda = \frac{-1/4 - \text{Im}(v_z(z))}{\|v_z + \Phi(\cdot; z)\|_{L^2(\partial D)}^2} \quad z \in D$$

Recall that $v_z$ is approximated by the Herglotz wave function $v_{g_z}$ where $g_z$ is the regularized solution of the far field equation.
Let $\lambda \in L_\infty(\Gamma_I)$. The electric total field $E := E^s + E^i$ satisfies

$$\nabla \times \nabla \times E - k^2 E = 0$$

in $\mathbb{R}^3 \setminus \overline{D}$

$$\nu \times E = 0$$

on $\Gamma_D$

$$\nu \times (\nabla \times E) - i\lambda(\nu \times E) \times \nu = 0$$

on $\Gamma_I$

$$\lim_{|x| \to \infty} (\nabla \times E^s \times x - ik|x|E^s) = 0$$

where $E^i(x) := ik(d \times p) \times d e^{ikx \cdot d}$
Far Field Operator

The scattered field \( E^s \) has the asymptotic behaviour

\[
E^s(x) = \frac{e^{ik|x|}}{|x|} \left\{ E_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right) \right\}
\]

as \( r = |x| \to \infty \), \( \hat{x} = x/r \), \( k \) is fixed and \( E_\infty(\hat{x}) = E_\infty(\hat{x}, d, p) \) for \( \hat{x}, d \in \Omega \) and \( p \in \mathbb{R}^3 \) is the far field pattern.

The far field operator \( F : L^2_t(\Omega) \to L^2_t(\Omega) \) is defined by

\[
(F g)(\hat{x}) := \int_\Omega E_\infty(\hat{x}, d, g(d)) \, ds(d).
\]

The far field equation is \( (F g)(\hat{x}) = E_{e,\infty}(\hat{x}, z, q) \).
Examples of Reconstruction


Perfectly conducting teapot, exact geometry
Examples of Reconstruction

Reconstruction for low frequency
Examples of Reconstruction

Reconstruction for intermediate frequency
Examples of Reconstruction

Reconstruction for high frequency
Examples of Reconstruction

The exact geometry

Impedance boundary condition with $\lambda = 1$ is the red region

Perfectly conducting boundary condition is the blue region
Examples of Reconstruction

The reconstructed partially coated airplane (wavelength=0.7)
Examples of Reconstruction

The perfectly conducting airplane

The imperfectly conducting airplane
Examples of Reconstruction

Colton-Monk (2006)

Reconstruction of fully coated two ball with $\lambda = 1$ and $k = 4$. 
Examples of Reconstruction

<table>
<thead>
<tr>
<th>Exact</th>
<th>Exact $\partial D$</th>
<th>LSM</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.104</td>
<td>0.11</td>
</tr>
<tr>
<td>1</td>
<td>0.99</td>
<td>0.98</td>
</tr>
<tr>
<td>1.22</td>
<td>1.21</td>
<td>1.18</td>
</tr>
<tr>
<td>2</td>
<td>1.97</td>
<td>1.46</td>
</tr>
</tbody>
</table>

Reconstruction of $\lambda$ for the two balls. Here $k = 4$. 
The Inverse Medium Problem

\( \nabla \cdot A(x) \nabla v + k^2 v = 0 \quad \text{in} \quad D \)

\( \Delta u + k^2 u = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \overline{D} \)

\( v - u = 0 \quad \text{on} \quad \partial D \)

\( \frac{\partial v}{\partial \nu_A} - \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial D \)

\( \lim_{r \to \infty} \sqrt{r} \left( \frac{\partial u^s}{\partial r} - i k u^s \right) = 0 \)

\( u(x) = u^s(x) + e^{i k x \cdot d}, \quad d \in \Omega := \{ x : |x| = 1 \}, \quad \frac{\partial v}{\partial \nu_A} := \nu \cdot A \nabla v \)

\( A \) is a symmetric, \( \Re (\bar{\xi} \cdot A \xi) \geq \gamma |\xi|^2 \) and \( \Re (\bar{\xi} \cdot A^{-1} \xi) \geq \gamma |\xi|^2 \), \( \Im (\bar{\xi} \cdot A \xi) \leq 0, \Im (\bar{\xi} \cdot A \xi) \leq 0. \)
The scattered field $u^s$ has the asymptotic behaviour

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}, d) + O\left(r^{-3/2}\right)$$

as $r \to \infty$ where $r = |x|$, $\hat{x} = x/r$, $k$ is fixed and $u_\infty$ is the far field pattern of the scattered field $u^s$.

The inverse scattering problem is to determine the support $D$ and the index of refraction $A$, from a knowledge of $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in \Omega$ (and possibly for a range of frequencies $k$).
Uniqueness Theorems

Theorem (Uniqueness of $D$) Hähner (2000): Assume that either
\[ \bar{\xi} \Re (A - I)\xi \geq \delta \|\xi\|^2 > 0 \] or
\[ \bar{\xi} \Re (I - A)\xi \geq \delta \|\xi\|^2 > 0 \] in $D$ for some $\delta$. Then, $D$ is uniquely determined by $u_{\infty}(\hat{x}, d)$ for $\hat{x}, d \in \Omega$ and a fixed value of the wave number $k$.

Uniqueness of $A$

- Gylys-Colwell (1996); If $A$ is a matrix (anisotropic case) it is known that $u_{\infty}(\hat{x}, d)$ for $\hat{x}, d \in \Omega$ does not uniquely determine $A$ even if it is known for an interval of values of $k$.

- If $A(x) = a(x)I$ (isotropic case) Gylys-Colwell (1996) has shown that $u_{\infty}(\hat{x}, d)$ for $\hat{x}, d \in \Omega$ for a fixed frequency uniquely determine $a(x)$ provided that $a(x)$ is sufficiently close to a constant.

Note that in $\mathbb{R}^3$ the assumption that $a(x)$ is sufficiently close to a constant is not needed; see Nachman (1987).
Non-uniqueness for anisotropic media
The Far Field Equation

\[(Fg)(\hat{x}) := \int_{\Omega} u_\infty(\hat{x}, d) g(d) ds(d) = \Phi_\infty(\hat{x}, z), \quad g \in L^2(\Omega), \quad z \in \mathbb{R}^2\]

For \(z\) in \(D\) the far field equation has a solution \(g\) iff there exist a solution \((v_z, w_z)\) of the interior transmission problem

\[
\nabla \cdot A \nabla w_z + k^2 w_z = 0 \quad \text{and} \quad \Delta v_z + k^2 v_z = 0 \quad \text{in} \quad D
\]

\[
w_z - v_z = \Phi(\cdot, z) \quad \text{on} \quad \partial D
\]

\[
\frac{\partial w_z}{\partial \nu_A} - \frac{\partial v_z}{\partial \nu} = \frac{\partial}{\partial \nu} \Phi(\cdot, z) \quad \text{on} \quad \partial D
\]

such that \(v_z\) is a Herglotz wave function \(v_g(x) := \int_{\Omega} e^{ikx \cdot d} g(d) ds(d)\).
ITP: What is known!

Cakoni-Colton-Haddar (2007): Assume that in $D$ for some $\delta > 0$ either

\[ \bar{\xi} \cdot \Re (A - I)\xi \geq \delta \|\xi\|^2 \quad \text{C1} \quad \text{or} \quad \bar{\xi} \cdot \Re (I - A)\xi \geq \delta \|\xi\|^2 \quad \text{C2} \]

- The interior transmission problem satisfies the Fredholm alternative in $H^1(D) \times H^1(D)$.
- If $\Im (\bar{\xi} \cdot A\xi) < 0$ in $D$ then there are no transmission eigenvalues.
- If $\Im (\bar{\xi} \cdot A\xi) = 0$ then the set of transmission eigenvalues is discrete.
- Any transmission eigenvalue $k > 0$ must satisfy

\[ k^2 \geq \frac{\lambda(D)}{\sup_D \|A^{-1}\|_2} \quad \text{if C1 holds} \quad \text{and} \quad k^2 > \lambda(D) \quad \text{if C2 holds}, \]

where $\lambda(D)$ is the first Dirichlet eigenvalue of $-\Delta$ on $D$. 
Remarks

**Definition:** The values of $k$ for which the homogeneous interior transmission problem (i.e. $\Phi(\cdot, z) = 0$) has a non trivial solution are called transmission eigenvalues.

**Open Problem** Do transmission eigenvalues exists?

The first result in this direction is due to Sylvester and Päivärinta (2007) for the case of $\Delta u + k^2 n(x)u = 0$. They have shown that transmission eigenvalues exist provided the contrast $n - 1 > 0$ is large enough.

**Remark:** If $k$ is a transmission eigenvalue and $v_z$ is a Herglotz wave function then the far field operator $F : L^2(\Omega) \rightarrow L^2(\Omega)$ is not injective with dense range, where $v_z, w_z$ is the non zero solution of the homogeneous interior transmission problem, i.e. for $\Phi(\cdot, z) = 0$. 
Solution of the Inverse Problem

The support $D$ can be determined by the linear sampling method.

What, if anything, can be said about $A$ from a knowledge of $u_\infty$?

Recall two results we have proven: Assume that $\text{Im} \left( \xi \cdot A \xi \right) = 0$

- Any transmission eigenvalue $k > 0$ must satisfies

$$k^2 \geq \frac{\lambda(D)}{\sup_D \| A^{-1} \|_2} \quad \text{if} \quad \| \text{Re} A^{-1} \|_2 \geq \delta > 1$$

where $\lambda(D)$ is the first eigenvalue of $-\Delta$ on $D$.

- If $k$ is a transmission eigenvalue and $v_z$ is a Herglotz wave function then the far field operator $F : L^2(\Omega) \to L^2(\Omega)$ is not injective with dense range.
Estimates for $A$

The **first result** provides an estimate for the 2-norm of $A$

Assume that $\|A^{-1}(x)\|_2 \geq \delta > 1$ for all $x \in D$ and some constant $\delta$. Then,

$$\sup_D \|A^{-1}\|_2 \geq \frac{\lambda(D)}{k^2}$$

where $k$ is the first transmission eigenvalue and $\lambda(D)$ is the first eigenvalue of $-\Delta$ on $D$.

The **second result** provides a way to compute the first transmission eigenvalue from the far field. In particular, the norm of the (regularized) solution to

$$(Fg)(\hat{x}) = \Phi_{\infty}(\hat{x}, z_0) \quad z_0 \in D$$

should be large for such value of $k$; Cakoni-Colton-Haddar (2007).
Transmission Eigenvalues

\[ D \text{ is the L-shape } = \{ [-0.5, 0.5] \times [-0.5, 0.5] \} \setminus \{ [0, 0.5] \times [0, 0.5] \}, \]
\[ A^{-1} = nI, \ n = 4 \text{ and } \eta = 0. \]
Numerical Examples

$D$ is the L-shape $= \{[-0.5, 0.5] \times [-0.5, 0.5] \} \setminus \{[0, 0.5] \times [0, 0.5] \}$, $A^{-1} = nI$, $\eta = 0$ and $\lambda(D) = 38.6$

<table>
<thead>
<tr>
<th>$n$</th>
<th>2.</th>
<th>3.</th>
<th>4.</th>
<th>6.</th>
<th>9.</th>
<th>12.</th>
<th>16.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_0$</td>
<td>15.5</td>
<td>8.1</td>
<td>6.3</td>
<td>4.5</td>
<td>3.3</td>
<td>2.8</td>
<td>2.3</td>
</tr>
<tr>
<td>$n_{\text{min}}$</td>
<td>0.2</td>
<td>0.6</td>
<td>1.</td>
<td>1.9</td>
<td>3.5</td>
<td>4.9</td>
<td>7.2</td>
</tr>
</tbody>
</table>

First transmission eigenvalues ($k_0$) and lower bounds of the index of refraction $A^{-1} = nI$
Qualitative Methods in Inverse Scattering Theory

An Introduction

Available online at springerlink.com

Springer