Triangulations, Heegaard splittings and hyperbolic geometry

The American Institute of Mathematics

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Corrections and new material are welcomed and can be sent to workshops@aimath.org

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A.1 Bachman, David

I am mostly interested in the stabilization problem. I now have examples of Heegaard splittings that require many stabilizations to become equivalent. However, these examples are all splittings of Haken manifolds. At the workshop I would like to explore the possibility of splittings of non-Haken 3-manifolds with high stabilization bounds.

A.2 Futer, David

I am interested in several open questions that relate triangulations, unknotting tunnels, and geometry. For example:

A. A genus-2 Heegaard splitting of a cusped hyperbolic manifold can be placed into almost normal form with respect to any triangulation. If we assume that the triangulation is geometrically canonical, when can the almost normal piece be an octagon? Typically, we would expect a tube, with the tunnel isotopic to an edge.
B. In all known examples, an unknotting tunnel is either isotopic to an edge of the canonical triangulation, or else dual to a face. Can more complicated behavior ever arise?
C. Cho and McCullough showed that every tunnel of every tunnel number one knot in S^3 can be *uniquely* obtained by an iterative sequence of simple moves. Is there a standard way to realize these moves by modifying an ideal triangulation of the knot complement? A positive answer to this question would help with the previous two problems.

A.3 Gordon, Cameron

I am interested in understanding the behavior of Heegaard genus under Dehn filling. Results of Moriah-Rubinstein and Rieck-Sedgwick show that the genus decreases by at most 1 outside a finite set S of filling slopes, and doesn't decrease at all outside the union of S and a finite union of lines. On the other hand, the case of exteriors of knots in S^3 shows that the genus can decrease arbitrarily under a single filling. In joint work with Ken Baker and John Luecke we have shown roughly that this cannot happen for non-integral Dehn fillings on knot exteriors. But in general very little seems to be known about the nature of the finite exceptional set S and what happens to Heegaard genus under fillings in that set.

I am also interested in learning about recent developments in the connections between Heegaard splittings and other aspects of 3-dimensional topology, such as the curve complex, hyperbolic geometry, and Heegaard Floer homology.

Finally, I am interested in hearing about Bachman's recent results on the stabilization problem.

A.4 Gueritaud, Francois

Here are a few open questions.

A. Positively–oriented ideal hyperbolic tetrahedra are parametrized by the cross-ratio of their four vertices, which is an element of the upper half-plane $\mathbb{H}$ of $\mathbb{C}$. Does there exist a compact subset K of $\mathbb{H}$ such that any ideal triangulation of any cusped finite-volume hyperbolic 3–manifold has at least one tetrahedron whose cross-ratio is in K?

B. More generally, does there exist a sequence $K_1 \subset K_2 \subset \dots$ of compacta of $\mathbb{H}$ such that any ideal triangulation as above, with N tetrahedra, has at least one tetrahedron in K_1, two tetrahedra in K_2, ..., N tetrahedra in K_N?

C. Let $ABCD$ be a tetrahedron. Identify the faces ABC and BCD (respecting the order of vertices) to obtain a solid torus T, with ∂T triangulated into two triangles. Perform a (non-backtracking) sequence of diagonal exchanges in ∂T, and realize each diagonal exchange by gluing a new tetrahedron to T by two of its faces. The resulting space is still a solid torus T', with boundary triangulation τ. Jaco and Rubinstein have conjectured that no other triangulation of T', inducing τ on the boundary, has fewer tetrahedra.

D. Let S be a closed surface, $f : S \to S$ a pseudo-Anosov, and $(g_t)_{t\in\mathbb{R}}$ the associated family of singular affine structures on S with $k\pi$–singularities (so the g_t's are push-forwards of each other by diagonal 2×2 matrices). For each t there is a natural affine "Delaunay" decomposition of (S, g_t) whose vertex set is the set X of singularities. Assume these decompositions are triangulations, related to one another by diagonal exchanges at singular values of t. This diagonal exchange pattern defines an ideal triangulation T of the $(S \smallsetminus X)$–bundle M with monodromy f. Does T admit positive angle structures? Can T be realized geometrically in the hyperbolic manifold M? Is T a Delaunay triangulation in the hyperbolic sense?

A.5 Hempel, John

My general interest is in studying closed, oriented 3-manifolds via their (Heegaard) splittings. In particular, I am interested in translating combinatorial properties of a description of a splitting, say by a (Heegaard) diagram, into topological/geometric properties of the underlying 3-manifold as well as ways to find the "simplest" diagram for a given splitting or the "simplest" splitting for a given manifold. Some particular questions:

1. In *3-manifolds as viewed fro the curve complex*, Topology(40), 2001, 631–657 I introduced the notion of *distance* for a splitting which has been useful in these contexts. But this *distance* is clearly a function of splittings – it reduces to zero on stabilization. It would be useful to have a related concept that is invariant under stabilization and so is intrinsic to the underlying manifold, Presumably this would be something like distance in the pants complex (of the splitting surface), but my efforts to make this precise leave many unsettled problems. Surely any such "generalized distance" would be all that much more difficult to compute (or estimate), but the existence of such could help settle some of the obvious unsettled questions related to the (ordinary) distance: like how must this distance compare for different irreducible splittings of the same manifold. It would surely relate to the "surgery distance" of the underlying manifold– the number of simple closed curves in S^3 one must surger to get the manifold.

2. I have, over several years, developed some inequalities giving lower bounds for the distance of a genus two splitting in terms of data from a diagram for the splitting . In particular a condition that implies distance at least 3 (and so the underlying manifold has a hyperbolic structure) is new and the others have a slight sharpening. I would like to discuss this material.

3. There are a number of reasons to believe that the case of genus two splittings is considerably more tractable than the general one– as evidenced by item 2 above and due, in large part, to a symmetry of their diagrams which fails for higher genus. I believe that this case should should studied intensively and that the number of parameters involved is small enough that computer studies could give meaningful insights. For example there are many different ways of measuring the complexity of a diagram. There is little evidence to indicate how far apart, in the space of handle slides, relative minima of (any of) these measures can occur. I have (slowly developing) plans for computerizing such problems. However I am confident that they would develop much faster if I shared my ideas with someone with more computational and programming skills.

A.6 Jaco, Bus

I am interested in triangulated Heegaard splittings. These were introduced by Hyam Rubinstein and myself in "Layered triangulations of 3-manifolds", http://front.math.ucdavis.edu/0603.5601, where the theory was developed for genus one but only introduced for higher genera Heegaard splittings. That work has been significally expanded upon in joint work with D. Farmer, J. Maher, D. McCullough and E. Sedgwick. It shows promise of being a very useful way of connecting special triangulations with Heegaard splittings and with the curve complex.

Another area is using triangulated Dehn fillings to investigate Heegaard splittings of Dehn fillings. Work in this direction was also introduced in the above referenced paper. Hyam and I are now looking at getting complete classifications for Dehn fillings obtained from some examples of knots in S^3.

Finally, Hyam, Joseph Maher and I have been looking at triangulations that are dual to one-sided incompressible Heegaard splittings (i.e., a one-sided incompressible surface in an orientable 3-manifold having the complement of a small open regular neighborhood a handlebody). These triangulations should be "efficient."

A.7 Johnson, Jesse

I am interested in ways of using one's knowledge of the geometry of a 3-manifold to determine information about its combinatorial structures, in particular its Heegaard splittings. This is, in some sense, the converse of the work currently being done by Namazi and a handful of others, who have found ways to construct a hyperbolic metric (or something very close to hyperbolic) on a 3-manifold from combinatorial information about a Heegaard splitting. I would like to find ways that the hyperbolic geometry can be used to construct Heegaard splittings, determine when two Heegaard surfaces are isotopic, or determine combinatorial information (such as the Hempel distance) about a given Heegaard splitting.

A.8 Li, Tao

I am particularly interested in discussions of Heegaard genus of toroidal 3-manifolds. The following 2 questions seems relevant.

1. What can one say about efficient triangulations of toroidal 3-manifolds? A very useful property of a 0-efficient triangulation of an atoroidal 3-manifold is that every normal torus bounds a solid torus. However, if the 3-manifold is toroidal, what can be said about the space of incompressible tori?

2. Let M be a 3-manifold with incompressible boundary and suppose M has more than one boundary component. Is there any "good" triangulations for M?

A.9 Manjarrez-Gutierrez, Fabiola

As a graduate student I have been working with Heegaard splittings and generalized Heegaard splittings as well. Specifically I am generalizing the concept of thin position for 3-manifolds, first introduced by Scharlemann and Thompson, to the new idea of "circular thin position". The idea is to look at a circular structure in the 3-manifold and study the level surfaces and find nice properties. So far I have obtained nice results for knot complements.

I expect to learn new 3-manifolds techniques in the workshop that will enlighten my research.

A.10 McCullough, Darryl

In recent work with Sangbum Cho, I have used the disk complex of the genus-2 handlebody to develop a theory of tunnel number 1 knots. Among the questions arising in this work are:

1. How does the disk complex sit in the curve complex of the boundary surface? One goal would be to obtain results about Hempel distance of tunnel number 1 knots.

2. The theory shows that every tunnel of every tunnel number 1 knot is obtained by a unique sequence of simple "cabling constructions." Can this be used to obtain information about the hyperbolic structures of the complements of tunnel number 1 knots?

In other recent work, with with David Farmer, Bus Jaco, Joseph Maher, and Eric Sedgwick, I have been studying minimal triangulations of handlebodies, and their use to construct 1-vertex triangulations of closed 3-manifolds. I hope to further understand this construction and its possible applications.

A.11 Milley, Peter

My main research interest is the classification of small-volume hyperbolic 3-manifolds (primarily using Mom-technology), and ways of measuring the "complexity" of hyperbolic 3-manifolds such as volume. I am interested in investigating what connections exists between Mom-technology and the other combinatorial descriptions of manifolds that are the subject of this workshop.

Specific questions that I would interested in pursuing include: are there any relationships between the "Mom number" of a manifold and the Heegaard genus, or the complexity of an efficient triangulation? Can efficient triangulations provide an efficient algorithm for detecting hyperbolicity and non-hyperbolicity in manifolds, and can Mom technology provide any insight into the efficiency of existing algorithms? More generally, what connections if any exists between efficient triangulations and volume?

A.12 Moriah, Yoav

Every closed orientable 3-dimensional manifold M admits a *Heegaard splitting*, i.e., a decomposition of M into two handlebodies / compression bodies V_1 and V_2 which meet along their common boundary Σ. This surface Σ is called a *Heegaard surface* and is usually considered only up to isotopy in M. In fact, every 3-manifold has infinitely many such splittings. We will refer to various properties of the collection of Heegaard splittings of a given manifold M as the *Heegaard structure* of M. We now define some further terminology.

A Heegaard surface Σ is called *reducible* if it contains an essential simple closed curve which bounds disks in both handlebodies (compression bodies). If the 2-sphere composed of the two disks bounds a 3-ball then the Heegaard splitting will be called *stabilized*. It will be called *weakly reducible* if it contains two disjoint essential simple closed curves where the first bounds a disk in V_1 and the second bounds in V_2. Otherwise it will be called *irreducible* or *strongly irreducible*, respectively.

A breakthrough in understanding Heegaard splittings was achieved in the work of Casson and Gordon (see [CG]). They showed that if a 3-manifold has a weakly reducible Heegaard splitting then either the splitting is reducible or the manifold is Haken. It follows, in particular, all non-stabilized Heegaard splittings of closed non-Haken irreducible 3-manifolds are strongly irreducible.

It turned out that manifolds with strongly irreducible Heegaard splittings are quite easy to come by. In fact it was proved by Casson-Gordon [CG] Kobayashi [Ko] and Lustig-Moriah [LM3] that there are lots of infinite families of manifolds each of which has strongly irreducible Heegaard splittings of arbitrary high genus.

All the known examples can be constructed as a Haken sum of one surface S and n copies of another surface K and are hence denoted by $S + nK$. It was proved by Moriah, Schleimer and Sedgwick (see [MSS]) that if there are infinitely many strongly irreducible Heegaard splittings of the form $S + nK$ then the surface K is incompressible. It was then proved by Li (see [Li]) that all manifolds with infinitely many strongly irreducible Heegaard splittings are Haken.

Results by Rubinstein-Scharlemann, Scharlemann-Tomova ([RS], [Sc], [ST]) and others provide strong evidence that strongly irreducible Heegaard surfaces behave in many ways like incompressible surfaces and induce similar properties in manifolds.

The conclusion we draw from the above discussion is that one should now focus on weakly reducible Heegaard splittings. These have been obtained as indued Heegaard splittings of Heegaard splittings of wide knots and links [**?**] or by amalgamating Heegaard splittings. In both cases the Heegaard splittings are of minimal genus. There is no known example of a weakly reducible, non-stabilized Heegaard splitting which is not of minimal genus. Finding these will be one of the goals of this project.

Another goal of this project is understanding how the genus behaves with respect to the operation of connected sum. It is well known (see [Ha]) that the genus of closed 3-manifolds is additive with respect to connected sum. i.e., if the genus of M_1 is g_1 and the genus of M_2 is g_2 then $g(M_1 \# M_2) = g(M_1) + g(M_2) = g_1 + g_2$. Hence the interesting questions about the behavior of genus with respect to connected sum will be about manifolds with boundary.

Motivated by the natural notion of *connected sum of knots in* S^3 one can generalize the operation of connected sum to 3-manifolds with a torus boundary as follows: Identify two incompressible annuli $A_1 \subset \partial M_1$ and $A_2 \subset \partial M_2$ to obtain a new manifold $M = M_1 \cup_{A_1 = A_2} M_2$ which also has a torus boundary. We will refer to this operation as *annulus sum*.

One can now ask how does the genus of M compare to the genus of M_1 and M_2? The special case of knots in S^3 and the operation of connected sums of knots has historically been the subject of much study but so far the situation is not completely understood.

When considering connected sums of knots in S^3 it is convenient to use the notion of unknotting tunnels. A *tunnel system* for an arbitrary knot $K \subset S^3$ is a collection of properly embedded arcs $t_1, \dots, t_n$ in $S^3 - N(K)$ so that $S^3 - N(K \cup t_1 \cup \cdots \cup t_n)$ is a handlebody.

Given a tunnel system for a knot $K \subset S^3$ note that the closure of $N(K \cup t_1 \cup \cdots \cup t_n)$ is always a handlebody denoted by V and the handlebody $S^3 - N(K \cup t_1 \cup \cdots \cup t_n)$ will be denoted by W. Hence any tunnel system corresponds in a natural way to a Heegaard splitting (V, W) of genus $n+1$ of $S^3 - N(K)$, where W is the handlebody and $V - N(K)$ is a compression body. For a given knot $K \subset S^3$ the smallest cardinality of any tunnel system is called the *tunnel number* of K and is denoted by $t(K)$. Note that $g(S^3 - N(K)) = t(K) + 1$

Let K_1 and K_2 be knots in S^3. There are three possibilities:

(i) $t(K_1 \# K_2) = t(K_1) + t(K_2) + 1$. i.e., $g = g_1 + g_2$ (see [MR] and [MSY]).
(ii) $t(K_1 \# K_2) = t(K_1) + t(K_2)$. i.e., $g = g_1 + g_2 - 1$ (see [LM3] and [Mr2])
(iii) $t(K_1 \# K_2) = t(K_1) + t(K_2) - n$, where $n \geq 1$. i.e., $g = g_1 + g_2 - (n+1)$, where $n \geq 2$ (see [Mo1] and [Ko]).

We will concentrate on case (i). It is a well known fact that for any two knots K_1 and K_2 in S^3, $t(K_1 \# K_2) \leq t(K_1) + t(K_2) + 1$ (see e.g. [MR]). A necessary condition for an equality is that in any Heegaard splitting for $S^3 - N(K_i)$ no meridian curve on $\partial(S^3 - N(K_i))$, for $i = 1, 2$, is isotopic to a primitive curve on the handlebody component W^i, of the Heegaard splitting (see [Mo1]). By a *primitive* curve on the boundary of a compression body H we mean a curve which intersects an essential disk D of H in a single point. It was a conjecture of Morimoto that it is also a sufficient condition i.e., $t(K_1 \# K_2) = t(K_1) + t(K_2) + 1$ if and only if there is no Heegaard splitting for $S^3 - N(K_i)$ in which a meridian curve is primitive. The conjecture, as is, was recently proved to be false by Kobayashiand Rieck [KR] . A modified conjecture, see Conjecture 7.14 in [Mr1], requires that the knots be *prime*. Trying to prove the modified one will also be one of the goals of the proposed research.

As for case (iii): There are examples by Morimoto where $t(K_1 \# K_2) = t(K_1) + t(K_2) - 1$ for two prime knots in S^3. The only known examples for which $t(K_1 \# K_2) < t(K_1) + t(K_2) - 1$ (see [Ko]) are such that both of K_1 and K_2 are composite knots i.e., connected sums of knots, iterated many times. It is unknown whether $t(K_1 \# K_2) < t(K_1) + t(K_2) - 1$ can occur if both knots are *prime*. We would like to investigate this question further.

A third aspect of this proposal addresses what might be considered the main difficulty in dealing with Heegaard splittings. It is the fact that a Heegaard splitting corresponds to a double coset $\mathcal{H}\phi\mathcal{H}$ of an element ϕ in the mapping class group $\mathcal{MCG}_g$ of a closed surface of genus $g \geq 2$, where $\mathcal{H}$ is the subgroup of surface homeomorphisms which extend to the a handlebody H via a properly chosen identification $\Sigma = \partial H$. This subgroup is not normal in $\mathcal{MCG}$, and it is not well understood at all. The geometric analogue of this problem is the absence of a canonical "coordinate system" that is, a preferred choice of disks which define the handle structure in the handlebodies respectively.

In recent work with M. Lustig we showed that if one imposes some conditions on the Heegaard splitting namely that they satisfy the Double Rectangle Condition (see [LM7]) then these Heegaard splittings have only a finite collection of complete decomposing systems of disks which satisfy this condition. As an immediate corollary we obtain that the manifolds which have Heegaard splittings satisfying the Double Rectangle Condition have a finite mapping class group. There is hope that this method will also yield results towards the homeomorphism problem.

The idea is to find combinatorial approximations on train tracks that will give the necessary complexity on curve systems carried by these train tracks so that the manifolds determined by gluing handlebodies via these curves will give control over the topology of the manifolds.

We will describe the objectives of this proposal by formulating a series of questions which we hope to answer.

Question 1. *Find an example of a* 3*-manifold with a non-stabilized weakly reducible Heegaard splitting ?*

A solution to this question will require techniques for detecting when a Heegaard splitting is reducible. Non exist so far. The only examples for non-stabilized weakly reducible Heegaard splittings are of minimal genus and hence they cannot be destabilized. The minimality is decided by algebraic techniques. This problem has importance beyond the immediate relevance to Heegaard structure. Finding techniques to decided when a Heegaard splitting is stabilized might have consequences with respect to the *virtually Haken conjecture*: It is known that any hyperbolic manifold with a strongly irreducible Heegaard splitting has finite covers with a weakly reducible Heegaard splitting. However these splittings might be stabilized. If one can rule out stabilization then one can apply the Casson-Gordon theorem to conclude that the cover is Haken.

A positive answer to Question 1 immediately gives rise to:

Question 2. *Does there exist a* 3*-manifold with non-stabilized weakly reducible Heegaard splitting of arbitrarily high genus?*

Such phenomena exist for strongly irreducible Heegaard splittings with interesting consequences for the manifolds. It would be interesting to find out what the implications for the topology follow from the existence of infinitely many weakly reducible Heegaard splittings.

This question relates to other issues involving stabilization. For example; It is known that Seifert fibered spaces have certain stabilized Heegaard splitting which can destabilize in two distinct ways to obtain two distinct Heegaard splittings. It might be possible to characterize the non-uniqueness of the destabilizing 2-spheres. This phenomenon is related to a question regarding the possible non-uniqueness of prime decompositions in the category of 3-manifolds with Heegaard splittings:

Question 3. *Is there a pair, M_1 and M_2, of manifolds and a Heegaard splitting (V, W) of $M = M_1 \# M_2$ which decomposes along two decomposing spheres which are isotopic in M, but which intersect $(V, W))$ in two different ways thus decomposing (V, W) into two different pairs of Heegaard splittings of M_1 and M_2?*

The following questions will deal with manifolds which are knots complements in S^3 and the operation of annulus sum (or connected sum of knots).

<u>Remark:</u> Given two knots $K_1, K_2 \subset S^3$ and Heegaard splittings (V^i, W^i), $i = 1, 2$, for their complements $S^3 - N(K_i), i = 1, 2$, then if a meridional curve in $\partial_- V^i$ is isotopic to a primitive

curve in W^i for at least one i then the two Heegaard splittings will *induce* a Heegaard splitting (V, W) of $S^3 - N(K_1 \# K_2)$ in a natural way (see [Mr2]). It is known (see [Mr2]) that (V, W) is weakly reducible even if both of $(V^i, W^i), i = 1, 2$, are strongly irreducible.

Question 4. *Let $K_1, K_2 \subset S^3$ be knots. Assume that the complements $S^3 - N(K_i), i = 1, 2$, have non-stabilized Heegaard splittings $(V^i, W^i), i = 1, 2$ which induce a Heegaard splitting (V, W) of $S^3 - N(K_1 \# K_2)$. Can (V, W) be a stabilized Heegaard splitting?*

In a slightly different flavor we ask:

Question 5. *Let $K_1, K_2 \subset S^3$ be* **prime** *knots with respective tunnel numbers $t(K_1)$ and $t(K_2)$. Show that a necessary and sufficient condition for the tunnel number to be super additive $t(K_1 \# K_2) = t(K_1) + t(K_2) + 1$ is that non of $S^3 - N(K_i), i = 1, 2$, have Heegaard splittings $(V^i, W^i), i = 1, 2$, where a meridian curve in $\partial N(K_i)$ is isotopic to a primitive curve on $W^i, i = 1, 2$.*

Remark: This statement is a modification of a conjecture of Morimoto's in [Mo2]. There is some evidence that the original conjecture, where the knots are not required to be prime, is false. Morimoto (see [Mo2]) has proved that if neither $S^3 - N(K_1)$ nor $S^3 - N(K_2)$ contain essential meridional surfaces or if $t(K_1) = t(K_2) = 1$ then the answer is yes.

Consider now a situation where the two knots $K_1, K_2 \subset S^3$ with respective tunnel numbers $t(K_1)$ and $t(K_2)$ satisfy $t(K_1 \# K_2) \leq t(K_1) + t(K_2) - 1$. It is known by work of Schultens and Morimoto that a necessary condition for this to occur is that at least one of $S^3 - N(K_i), i = 1, 2$, contain essential meridional surfaces. It is known by work of the author that there are examples of knots containing such surfaces where the tunnel number does not decrease. This implies that the surfaces one needs to look for are indeed very special. We state this as the following question:

Question 6. *Find a necessary and sufficient condition on the essential meridional surfaces in $S^3 - N(K_i), i = 1, 2$, so that $t(K_1 \# K_2) \leq t(K_1) + t(K_2) - 1$.*

Regarding the third proposed aspect of this project there are two directions in which to I would like to proceed. The first is:

Question 7. *Given a Heegaard splitting for a manifold M. Is there an effective algorithm to find all of the complete decomposing systems that satisfy the Double Rectangle Condition?*

Question 8. *Find an effective algorithm to decide if M has a Heegaard splitting which satisfies the Double Rectangle Condition.*

The second direction has do do with the Double Rectangle Condition itself. This condition is a slight generalization of the Casson-Gordon Rectangle Condition (first used in [CG]). Although there are plenty (infinitely many) of manifolds that have Heegaard splittings which satisfy this condition it is somewhat "unnatural". Finding a more natural condition would be a big improvement in the theory and possibly yield insight into the Heegard structure of the manifold. Thus we can state:

Question 9. *Find a "natural" condition to replace the Double Rectangle Condition.*

The main tool of attacking questions 3 - 6 is to use the existence of a decomposing annulus in a manifold which is an annulus sum (connected sum of knots in S^3) to try and get information about the Heegaard splittings. If we assume that the knots are prime (or that the manifolds are un-annular) then this annulus is unique up to isotopy. We can now proceed in one of two directions:

The first is the approach used by M. Scharlemann and J. Schultens in [SS] [SS1] and also K. Morimoto and J. Schultens in [MS]. One can construct a "generalized Heegaard splitting" from the given Heegaard splitting. In the case of a strongly irreducible Heegaard splitting, this is just the original Heegaard splitting. In the case of a weakly reducible Heegaard splitting, one constructs a collection of incompressible surfaces that separate the manifold into submanifolds, each of which inherits a strongly irreducible Heegaard splitting. One can then isotope the decomposing annulus so that it intersects the Heegaard splitting in a collection of concentric curves, thus dividing the annulus into sub-annuli so that adjacent sub-annuli are contained in distinct compression bodies of the generalized Heegaard splitting. One can study this collection of annuli in the handlebodies (compression bodies) and can draw quite powerful conclusions (see again e.g. [SS] [SS1] and [MS]).

The second approach is the one used by the author (see [Mr2]) where the intersection of the decomposing annulus with a set of meridian disks for the handlebodies (compression bodies) is studied directly using some earlier work of Morimoto and Ochiai (see [Mo3], [Oc]). Since the intersection is very restrictive one can also obtain valuable information about the Heegaard splittings.

For Question 7 the idea is to carefully analyze the method used by the author and Martin Lustig in [LM7]. The feeling is that there is sufficient power in that method to shed some light onto the above question. The following is a sketch of the approach:

Start with any pair of complete decomposing disk systems in the handlebodies H_1 and H_2, and apply Hempel's algorithm to transform it into pairs decomposing systems $\mathcal{D}_1 \subset H_1$ and $\mathcal{D}_2 \subset H_2$ that have no waves with respect to each other. Check whether every disk of $\mathcal{D}_1$ intersects every disk of $\mathcal{D}_2$. If not, it should follow from one of our lemmas of [LM7] that there can not exist any pairs of disk systems $\mathcal{D}'_1, \mathcal{D}'_2$ which satisfy the rectangle condition.

Otherwise, we proceed to define arc types on the boundary of the solid pairs of pants B_k from the collection $H_1 - \mathcal{D}_1$, and thus one can construct all of the finitely many disk types. It follows that, if there exists any pair $\mathcal{D}'_1, \mathcal{D}'_2$ which satisfies the rectangle condition, each disk of $\mathcal{D}'_1$ is made out of these disk types.

It suffices now to compute the bound $\mathcal{A}$ defined in [LM7], in order to be able to construct the finite collection of disk systems that must contain any system $\mathcal{D}'_1 \subset H_1$ which satisfies the double rectangle condition with respect to some disk system in H_2. For this purpose we follow the method designed in sections 4 and 5 of [LM7]] to determine all of the thick and thin parts and regions. We count the number of thick regions, their maximal diameter d, and the number of their complementary components, as well as of their $(c + 2d)$-neighborhoods: Here c is the minimal number of intersections of any curve from $\mathcal{D}_2$ with the system $\mathcal{D}_1$. These constants suffice to follow through the proof of our main Proposition of [LM7] and to calculate an upper bound $\mathcal{A}$ as stated there.

Thus we obtain finitely many candidates for $\mathcal{D}'_1$, and similarly finitely many candidates for $\mathcal{D}'_2$, and it is a finite procedure, to check which of the possible pairs $\mathcal{D}'_1, \mathcal{D}'_2$ which satisfy the double rectangle condition. It follows directly from [LM7] that in this way we have found indeed all pairs of completes disk systems in H_1 and H_2 that satisfy the double rectangle condition. In particular, if we didn't find any, there is no such pair.

The two other question namely 8 and 9 seem more difficult. However the ideas of [LM7] were never before explored and the hope is that as more thought goes into them progress will be achieved.

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A.13 Ni, Yi

I am interested in the problem whether there is a connection between Heegaard Floer homology and the classical theory of Heegaard splittings, (say, Casson-Gordon's theorem, generalized Heegaard splittings, curve complex...).

A.14 Rieck, Yoav

My personal interest in this workshop is the study of Heegaard splitting and its relationship to other techniques in 3-manifold theory, both the purpose of applying new techniques for questions about Heegaard splittings (for example, finiteness) and for the purpose of using Heegaard splittings (as well as other techniques) for studying 3-manifolds, and in particular algorithms on 3-manifolds.

A.15 Rubinstein, J. Hyam

Project 1 (with Jesse Johson) Given a Heegaard splitting (M, S) , an automorphism is a homeomorphism $\phi : M$ which takes S to itself. These form a group and by restricting to a subgroup of index two, we can assume that the handlebodies on either side of S are preserved by ϕ. Our main results are classifications of the types of M and S which can admit periodic or reducible automorphisms. In particular we obtain bounds on the Hempel distance of S. A key question is whether there are instances where all automorphisms are pseudo Anosov.

Project 2 (with Loretta Bartolini). A one sided Heegaard splitting is a non orientable surface K with complement a handlebody and is obtained from any Z_2 homology class of a closed orientable 3-manifold. We establish a version of the key Casson-Gordon result - with incompressibility being the substitute for strong irreducibility. Since Haken's lemma does not apply to one sided surfaces, a completely new argument is required. To summarise - we show that if a one sided Heegaard splitting K is compressible, then either the 3-manifold is Haken or else the splitting is stabilised, ie has a trivial handle. We also find an interesting relationship between incompressible one sided splittings and spanning surfaces for knots and links. For the case of the figure 8 knot space, we prove that any $(2n, 1)$ filling has a unique incompressible one sided splitting, for $|n| \geq 3$ whereas $(8, 3)$ filling has three different incompressible one sided splittings. So one has an interesting comparison with ordinary Heegaard splittings. To our knowledge, no hyperbolic 3-manifolds are known for which all irreducible Heegaard splittings are classified.

Project 3 (with Stephan Tillmann and Bus Jaco) We investigate some 3-manifolds for which the minimal triangulations can be classified. This is an interesting challenge. At present, our methods apply to a special class of lens spaces, those of type $L(2n, 1)$ and probably related Seifert fibered spaces. The key idea is to use the canonical one sided normal surface associated to a Z_2 homology class and the special properties of layered triangulations. We hope to extend the methods to look at Dehn fillings on the figure 8 knot space but this looks very challenging at present. A key question is whether minimal triangulations of large Dehn fillings should look like large layered solid tori attached to small triangulations of the thick part, in the sense of Margulis.

A.16 Scharlemann, Martin

I'm particularly interested in

*understanding the hyperbolic viewpoint on Heegaard splittings, and look forward to hearing from Jeff Brock, Juan Souto, Hossein Namazi.

*picking up what I can on new results in Heegaard Floer homology, e. g. the work of Yi Ni

*learning what has been happening on classic problems in combinatorial Heegaard theory, e. g. the Bachman + Qiu solutions to the Gordon conjecture.

A.17 Schleimer, Saul

Interests:

Triangulations of three-manifolds arising from combinatorial data (Heegaard splittings, knot tunnels, surface automorphisms) or from geometric data (hyperbolic structures).

Automorphism groups of highly stabilized Heegaard splittings: for example the Goeritz group.

The curve complex and algorithms to compute the Hempel distance.

A.18 Schultens, Jennifer

In recent years there has been a flurry of activity in the area of 3-manifolds that relates combinatorial structures, such as triangulations and Heegaard splittings, to more geometric structures, such as hyperbolic geometry. My intention is to revisit some of these combinatorial structures and delve more deeply into their refinements, such as efficient triangulations and the curve complex view of Heegaard splittings. My goal is to see how these more refined notions help to illuminate the relation of the combinatorial structures to hyperbolic geometry.

A.19 Stevens, Alice

I am student interested in Heegaard splittings of 3-manifolds. I would like to learn more about these new methods for decomposing 3-manifolds. I believe the lectures on background material will be particularly useful for me.

A.20 Tomova, Maggy

My recent work has been focused on Heegaard splittings and distance in the curve complex. Triangulations are closely related to Heegaard splittings so I'd like to learn more about triangulations, and more specifically, efficient triangulations and what information they carry.

Ways to figure out the distance of a Heegaard splitting would also be of great interest to me.

A.21 Wilson, Robin

Given a normal surface or an almost normal surface in a triangulation of a 3manifold is there an algorithm to determine whether or not the surface is a Heegaard surface?