Remarks on the Tate Conjecture for beginners

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The pair of conjectures of Tate, called **T1** and **T2** below, are of great importance. They provide analogues of two other famous conjectures, namely the Hodge Conjecture, which is a prediction in Complex Geometry, and the Birch and Swinnerton-Dyer Conjecture, which is a centrepiece of Number theory. This dual aspect is what makes the Tate conjecture central and also very difficult to solve in general. After stating the conjectures, I will discuss them for the special case of a product of an elliptic curve E with itself.

First the *geometric aspect*. Let X be a smooth projective variety over the complex numbers $\mathbb{C}$. The simplest example of X is the projective space $\mathbb{P}^N$, whose points are represented by non-zero vectors $x = (x_0, x_1, \ldots, x_N)$ in $\mathbb{C}^N$ where x is identified with λx for any scalar λ; it is also the image of the (real) $(2N-1)$-dimensional sphere S^{2N-1} under the antipodal map. More generally, X is cut out in $\mathbb{P}^N$ by a finite number of homogeneous polynomials $f_i(x), 1 \leq i \leq r$. In other words, X is the set of simultaneous solutions, up to scaling, of these polynomials. We also want it to be smooth, i.e., without any corners or folds (which can be checked by a Jacobian condition). Then, under the complex topology, X becomes a compact complex manifold, equipped with a K̈hler metric, of certain dimension d. When the polynomials form an independent system, d would be $N - r$. When $r = 1$ we get a hypersurface (*prime divisor*), and this is already interesting for $N = 2$, as X would be a complex algebraic curve. (It is a real surface!) Of fundamental importance are the (singular) cohomology groups $H^j(X, \mathbb{C})$ of dimension b_j, which vanish if $j \notin [0, 2d]$, with $b_0 = b_{2d} = 1$. There is a duality relating H^j to H^{2d-j}, and moreover, there is a Hodge decomposition $H^j(X, \mathbb{C}) = \oplus_{p=0}^{j} H^{p,j-p}(X)$, where $H^{p,q}(X)$ is generated by differential forms ω which are locally of the form $\xi dz_I \wedge \overline{d}z_J$ with $|I| = p, |J| = q$; in other words this differential form has p holomorphic wedge components and q anti-holomorphic ones. A fundamental problem is to understand the (closed) subvarieties Z in X. If Z is of codimension m in X, then we can take a $(2d-m, 2d-m)$-form ω on X and integrate it over Z. This defines a linear form on $H^{2d-m,2d-m}(X)$, and hence a class $c\ell(Z)$ in $H^{m,m}(X)$, which also lies in $H^{2m}(X,\mathbb{Q})$, the $\mathbb{Q}$-subspace of $H^{2m}(X,\mathbb{C})$ consisting of differentials with *rational* periods. One calls any element of $H^{m,m}(X) \cap H^{2m}(X,\mathbb{Q})$ a *Hodge cycle* of codimension m. A natural question which arises is whether all such Hodge cycles are rational linear combinations of such $c\ell(Z)$'s, and the celebrated Hodge conjecture asserts that the answer is yes. This is a sort of a rigidity conjecture, as it asserts that all the rational cohomology classes of type (m,m), which are *a priori* flabby topological constructs, are all algebraic. It is instructive to look at the special case when $X = E \times E$, where E is an elliptic curve given by the equation $y^2 = f(x)$, where $f(x)$ is a cubic polynomial with three distinct roots. The complex solutions making up the geometric points of E form a torus $\mathbb{C}/L$, where L is a lattice; if $\mathfrak{P}_L(z)$ denotes the Weierstrass P-function $\frac{1}{z^2} + \sum\limits_{\lambda \in L}\left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right)$, then this identification is seen by setting $x = \mathfrak{P}_L(z)$ and $y = \mathfrak{P}'_L(z)$. The fundamental group $\pi_1(E)$, and its abelian quotient $H_1(E,\mathbb{Z})$ are both just the lattice L, which is of rank 2. So $H^1(E,\mathbb{C})$ is 2-dimensional, with $H^{1,0}(E)$, resp. $H^{0,1}(E)$, being generated by $dz(= dx/y)$, resp. $\overline{dz}$. Also, $H^0(E,\mathbb{C}) = \mathbb{C}$ and $H^2(E,\mathbb{C}) = H^{1,1}(E) = \mathbb{C}$ (because there is no holomorphic, or anti-holomorphic 2-form on a compact Riemann surface). By Künneth, we have (since X is a product $E \times E$)

$$(1) \quad H^2(X) = (H^0(E) \otimes H^2(E)) \oplus (H^1(E) \otimes H^1(E)) \oplus (H^2(E) \otimes H^0(E))$$

The first and the last groups on the right are both one-dimensional and of Hodge type $1,1$, and so give Hodge cycles. The Hodge conjecture are OK for these (trivial) classes because they can be represented by the hypersurfaces (*divisors*) $x_0 \times E$ and $E \times x_0$, where x_0 is any fixed point on E. The middle group on the right of (1), which is of dimension 4, is much more interesting, and we get

$$(2) \quad H^1(E) \otimes H^1(E) \simeq (H^{1,0}(E)^{\otimes 2}) \oplus (H^{1,0}(E) \otimes H^{0,1}(E))^{\oplus 2} \oplus (H^{0,1}(E)^{\otimes 2}).$$

The first, resp. last, group on the right of (2) is one-dimensional and purely of type $(2,0)$, resp. $(0,2)$, so there are no Hodge classes coming from either of them. The middle group on the right of (2), call it W, is of dimension

2 and purely of type $(1,1)$. So the number of independent Hodge classes, call it r_{Hg}, is at most 2. It is at least 1, because W contains the class of the diagonal curve $\Delta = \{(x,x)|x \in E\}$. If r_{Hg} is 1, we are done, so we may assume that it is 2. Then there is an extra rational class of type $(1,1)$. Put another way, a Hodge class in W, or rather an integral multiple of it, corresponds to a homomorphism $E \to E$, and Δ identifies with the identity homomorphism. If we have an independent Hodge class, then it will need to correspond to an endomorphism φ of E which is not multiplication by some n. In other words, E will have complex multiplication (CM, for short) by (an order R in an) imaginary quadratic field K. The desired cycle Z is then just the graph of such a complex multiplication. This gives an explicit proof of the Hodge conjecture in this special case.

Now onto Tate. Given any X as above, since it is defined by the simultaneous vanishing of a finite set of homogeneous polynomials f_i, we can find a finitely generated field k over which X is defined. (k is the field obtained by adjoining to $\mathbb{Q}$ all the coefficients of all the f_i.) In other words, if σ is an automorphism of $\mathbb{C}$, we can consider the variety X^σ obtained by the zeros of $\{f_i^\sigma\}$, which will be the same variety if σ fixes k pointwise. Denote by $\overline{k}$ the algebraic closure of k in $\mathbb{C}$ and by G the Galois group of $\overline{k}/k$. Then G acts on the $\overline{k}$-points of X, but it does not act on the singular cohomology groups $H^j(X,\mathbb{Q})$, except for the identity and complex conjugation. Because of this, we can keep track of which cycles are defined over k (or a finite extension field of k). Thankfully, the situation becomes quite good if we pick a prime ℓ and replace the $H^j(X)$ by their ℓ-adic analogues $H^j_\ell(X)$, which are $\mathbb{Q}_\ell$-vector spaces of the same dimension, and are moreover acted upon by G. The G-action replaces the Hodge structure (over $\mathbb{C}$), and a k-rational *Tate cycle* in $H^{2m}_\ell(X)$ is a class η which is quasi-invariant under G according to the *Tate character* $\chi_\ell(m)$, which is described by the m-th power of the character by which G acts on the ℓ-power roots of unity in $\overline{k}$. The conjecture **T1** of Tate asserts that every such Tate cycle is represented by a k-rational $\mathbb{Q}_\ell$-linear combination of classes $c\ell_\ell(Z)$ of codimension m subvarieties Z. (Here k-rational does not mean that each Z is defined over k; for example, if Z is defined over a quadratic extension k' of k, then $Z + Z^\theta$ is k-rational, where θ is the non-trivial automorphism of k.) One write $r_{\ell,k}$, resp. $r_{\mathrm{alg},k}$, the number of independent Tate classes, resp. algebraic classes, and the "numerical" version of **T1** is that $r_{\ell,k} = r_{\mathrm{alg},k}$. If **T1** holds for some $k_1 \supset k$, then it holds for all k' in between k_1 and k. Moreover, all the Tate classes are defined over a finite extension k_1. It is expected that if **T1** holds for X for large enough k, then the Hodge conjecture also holds for X over $\mathbb{C}$. This is known to be true for an important class of X, called abelian varieties. In fact, if X is an abelian variety of CM type, then the Hodge and Tate conjectures are equivalent for X.

Now let us look at **T1** for the case $X = E \times E$, where E is defined over k. The Künneth formula works in the ℓ-adic setting as well, and making use of the trivial divisors $x_0 \times E$ and $E \times x_0$, for a k-rational point x_0, for example the origin, on E, we see that the key is to understand the Tate classes in the 4-dimensional ℓ-adic vector space

$$H^1_\ell(E) \otimes H^1_\ell(E).$$

The Tate classes (over k) can be identified with the $\mathbb{Q}_\ell$-endomorphisms ψ_ℓ of the 2-dimensional space $H^1_\ell(E)$ which commute with the action of G. By a deep theorem of Faltings, every such ψ_ℓ is a $\mathbb{Q}_\ell$-linear combination of honest endomorphisms φ of the elliptic curve E, which are moreover rational over k. In particular, the number satisfies (as in the Hodge case) $1 \leq r_{\ell,k} \leq 2$, and it equals 2 iff E has complex multiplication defined over k. In any case, it is also $r_{\mathrm{alg},k}$.

Now onto the L-function variant **T2**. It has been known for over a hundred years that every time a zeta or L-function $\sum_{n\geq 1} a_n n^{-s}$ has a pole (or zero), it often says something significant arithmetically. Suppose X is a variety defined over a number field k (think $k = \mathbb{Q}$ in the beginning), which comes with the G-representations $H^j_\ell(X)$ (with $G = \mathrm{Gal}(\overline{k}/k)$). This gives rise to an L-function $L^j(s, X)$, which has an Euler product, convergent (by Deligne's proof of the Weil conjectures) in the right half plane $\Re(s) > j/2+1$. It is expected to admit a meromorphic continuation to the whole s-plane and a functional equation relating s to $j+1-s$. The Birch and Swinnerton-Dyer conjecture is concerned with the central point $s = 1$ when $j = 1$, predicting that the order of zero at this point is the same as the rank of the k-rational points of the Picard variety of X; in the important special case when X is an elliptic curve E, the Picard variety is just E itself. The Tate conjecture **T2** is concerned with the (near-central) *Tate point* $s = m + 1$ when $j = 2m$. It asserts that the order of pole of $L^{2m}(s, X)$ at this point is the rank $r_{\mathrm{alg},k}$ of the k-rational algebraic cycles on X of codimension m. If we believe in the Langlands philosophy furnishing the *modularity* of any such $H^{2m}(X)$, then one can prove that the conjectures **T1** and **T2** are equivalent for X over number fields k, the reason being that the L-functions of non-trivial cusp forms on $\mathrm{GL}(n)$ are invertible at the critical edge.

Let u now look at **T2** for $m = 1$ and $X = E \times E$, for an elliptic curve E over $\mathbb{Q}$. By recent modularity results of Wiles, et al, we know that E is modular, and so $L^1(s, E) = L(s, f)$, for a cusp form $f = \sum_{n \geq 1} a_n q^n$, $q = e^{2\pi i z}$, of weight 2, trivial character, and level equal to the conductor N of E. One sees that the interesting (degree 4) part of $L^2(s, X)$ is the Rankin-Selberg L-function $L(s, f \times f)$, given essentially by the Dirichlet series $\sum_{n \geq 1} a_n^2 n^{-s}$. This L-function has a simple pole at the edge $s = 2$, which is accounted for by the cycle Δ on $E \times E$. If E has no CM, f will not be dihedral and this simple pole situation is expected to persist over any number field k, which can be verified if k is solvable. Suppose E has CM. Then f is associated to a (weight one) Hecke character χ of a quadratic imaginary field K, and the Rankin-Selberg L-function acquires another pole over k iff $k \supset K$, and the corresponding cycle is given by the graph of such a complex multiplication.

If one works out what **T2** implies for all the powers E^n of an elliptic curve E over $\mathbb{Q}$, then one comes to the conjecture that for each $m \geq 1$, the symmetric $2m$-th power L-function of E (or of f) has no zero (or pole) at the edge, which is $s = 1$ in the unitary normalization. If this holds on the whole line $\Re(s) = 1$, then the Sato-Tate conjecture will follow for non-CM E. This is what Taylor and his collaborators have proved when E has multiplicative reduction at a prime.

The Tate conjectures work over arbitrary finitely generated fields k, in particular when k is finite. Now suppose X is an elliptic surface over $k = \mathbb{F}_q$, so that X is a fibres over a curve C such that outside a finite number of points, the fibre over $s \in C$ is an elliptic curve. If F is the function field of C over k, then it is an analogue of the rationals, and moreover, the generic fibre is an elliptic curve E over F. The Tate conjecture **T2** for X is essentially equivalent to the Birch and Swinnerton-Dyer conjecture for E over F. This is another nice connection.