Selfsimilar groups and conformal dynamics
Problem List
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1 Self-similar groups and associated objects

We use terminology from the book [Nek05]. A group G is called *self-similar* if it admits a self-similar action.

1. It seems quite difficult to show whether a given group is self-similar.

   (a) Are Gromov hyperbolic groups self-similar?

   (b) Find obstructions to selfsimilarity.

   (c) Is $SL_3(\mathbb{Z})$ self-similar?

   (It is known that the free groups F_n are self-similar for every n.)

2. Can one compute the topological dimension of the limit space of a contracting self-similar group?

   An upper bound is $|\mathcal{N}| - 1$, where $\mathcal{N}$ is the nucleus of the action. This upper bound is far from sharp even in the simplest cases. For example, for the standard adding machine the obtained upper bound is 2, one more than the dimension of the circle.

3. Can one compute the conformal dimension of the limit space of a contracting self-similar action?

   The limit space can be realized as the boundary of a Gromov hyperbolic one-complex $\Sigma = \Sigma(G, S)$ where S is a set of generators. The quasi-isometry type of Σ depends only on the group action G. General principles then imply that the Gromov boundary $\partial\Sigma$ (called the *limit space* of the action) inherits a preferred set of metrics depending only on the action. This set of metrics is called the *conformal gauge* of $\partial\Sigma$. The *conformal dimension* of $\partial\Sigma$ is the infimum of the Hausdorff dimension over all metrics in $\mathcal{G}(\partial\Sigma)$ and is therefore a numerical invariant of the action. For more details see the preprint [HP06].

   (a) How is this invariant related to other numerical invariants of the self-similar action, e.g. contraction coefficients and growth of Schreier graphs of orbits on the boundary of the tree?

   In certain cases these quantities can be computed explicitly. Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be the map on the n-torus induced by a semisimple linear map with eigenvalues $1 < \lambda_1, \lambda_2, \ldots, \lambda_n$. Associated to this dynamical system is a self-similar action of $\mathbb{Z}^n$ (the standard action of the associated iterated monodromy group). It is known that the conformal dimension of the limit space associated to this action is $\frac{1}{\log \lambda_1} \sum_i \log \lambda_i = \log n / \log \lambda_1$. In this case, the growth of orbits is of degree n.

(b) Is the contracting coefficient of a contracting selfsimilar group always an algebraic number?

4. Is every contracting selfsimilar group amenable?

   Bartholdi, Kaimanovich, Nekrashevych, and Virag [BKNV06] have announced that all bounded automaton groups are amenable. It is proved in [Nek07] that contracting groups have no free subgroups.

5. Does there exist a self-similar group which is non-amenable but does not contain a non-commutative free subgroup? This may give an example which is much simpler than the existing ones.

6. Does every self-similar group act amenably on some compact space?

7. Which self-similar groups admit a finite L-presentation? Is it true that all contracting groups have finite L-presentation? Which groups in these classes have a finite ascending L-presentation? For definitions of L-presentations see [Lys85, Gri98, Bar03, BGŠ03].

8. What are possible Hausdorff dimensions of the closures of self-similar (contracting) groups in the automorphism group of the rooted tree? See some interesting examples in [Šun06].

9. What are the possible growth types of Schreier graphs of the action of a self-similar group on an orbit on the boundary of the tree?

   It is known that the boundary Schreier graphs of contracting groups have polynomial growth. What are the possible degrees?

2 Algorithmic problems

Fix a finite alphabet X of cardinality d. In this section "given a self-similar group" means that a recursive definition of a finite set of generators is given in the form $g_i(xw) = yh_{i,x}(w)$, where $h_{i,x}$ are group words in the generators, $x, y \in X$ and the word $w \in X^*$ is arbitrary.

An extensive list of algorithmic problems related to groups generated by automata is collected in [GNS00].

1. Does there exist an algorithm, which given a self-similar group, decides if it is contracting? (There exists an algorithm which stops if and only if the group is contracting.)

2. Do there exist algorithms which given a self-similar group decides if this group is

   (a) level-transitive,

(b) branch,

(c) weakly branch,

(d) just infinite,

(e) torsion,

(f) torsion free,

(g) free,

(h) of intermediate growth,

(i) of exponential growth,

(j) amenable?

One may of course pose the same questions for the class of contracting groups (except for (g), which is impossible for contracting groups and possibly (j), since we suspect that they all are amenable).

3. Do there exist algorithms which, given a pair of self-similar groups, decides if they are (i) isomorphic, (ii) conjugate in $\mathrm{Aut}(T)$? Obviously (ii) implies (i) but there are examples for which (i) does not imply (ii).

4. Find an algorithm which for a given element of a self-similar group, (i) decides if the order is finite; (ii) computes its order.

5. What can one say about the membership problem for self-similar groups? In particular, is the membership problem decidable for contracting groups? The membership problem is decidable for the Grigorchuk group $\mathcal{G}$, but false for some automaton groups. Conjugacy separability for $\mathcal{G}$ was proved by Y. Leonov [Leo98] (see also Zalesski and Wilson in [WZ97]), and subgroup separability for $\mathcal{G}$ was proved by Wilson [GW03].

6. Do there exist self-similar groups with unsolvable word problem? What are the possible complexities of the word problem for self-similar groups? Contracting groups have word problem of polynomial complexity (see Proposition 2.13.10 of [Nek05] and Chapter 3 of [BGŠ03]). Do there exist automaton groups whose word problem has exponential complexity?

7. Do there exist self-similar groups with undecidable conjugacy problem? All known examples of self-similar groups with solvable word problem also have solvable conjugacy problem. So can there exist a contracting self-similar group with undecidable conjugacy problem?

8. Is the isomorphism problem decidable in the class of contracting groups? That is, given two contracting groups, can one decide if they are isomorphic as abstract groups?

9. Does there exist an algorithm which for given two self-similar (or contracting) groups decides if these groups are equivalent, i.e., if the associated permutational bimodules are isomorphic?

10. The Grigorchuk group $\mathcal{G}$:

    (a) Find an algorithm for solving finite systems of equations in $\mathcal{G}$.

    (b) Is the elementary theory of $\mathcal{G}$ decidable?

    (c) To what type of languages does the language of geodesics in $\mathcal{G}$ belong?

    (d) $\mathcal{G}$ has a finite L-presentation. What is the structure of Van-Kampen diagrams of $\mathcal{G}$ over its L-presentation?

11. Do all contracting self-similar groups have a finite L-presentation?

3 Iterated Monodromy Groups

1. Describe automata generating iterated monodromy groups of post-critically finite rational functions. Find a nice class of automata such that the IMG of every post-critically finite rational function is generated by an automaton in the class.

   The *kneading automata* of [Nek05] provide such a class for the collection of post-critically finite polynomials. The difference between the polynomial and the rational case is the fact that polynomials admit a succinct combinatorial description with a canonical normal form, while no such description or normal form is available for general rational functions.

2. Find an algorithm which, for a given automaton (or wreath recursion) generating the IMG of a Thurston map (i.e., a post-critically finite branched self-covering of the sphere), decides whether it is equivalent to a rational function (i.e., conjugate up to homotopies relative to the post-critical set). This would give an algorithmic version of the Thurston's theorem on realizability of Thurston maps by rational functions. For more on Thurston maps and Thurston's theorem see [DH93].

3. Does there exist an algorithm which, given two IMGs of Thurston maps, decides if these maps are combinatorially equivalent? A general principle in dynamics is that suitably expanding systems are determined by a finite amount of data. Heuristically, one might expect the answer to this question to be yes for the class of Thurston maps which are (orbifold) expanding. See an example of analyzis of combinatorial classes in [BN06].

4. When is the IMG of a postcritically finite polynomial of intermediate growth?

   Bartholdi, Grigorchuk and Żuk showed that IMG(z^2-1) contains a free monoid on two symbols and thus has exponential growth, see [GŻ02]. Bux and Perez in [BP06] showed that $\mathrm{IMG}(z^2+i)$ has intermediate growth. The tuning [Dou87] of $f(z) = z^2-1$ by z^2+i results in a quadratic polynomial $g(z) = z^2 + c$ with dendrite Julia set J_c. There is a natural semiconjugacy $J_f \to J_g$. The functorial nature of the assignment of IMGs implies that there is an embedding $\mathrm{IMG}(f) \hookrightarrow \mathrm{IMG}(g)$ and hence that $\mathrm{IMG}(g)$ has exponential growth. See [Nek06], where functoriality of the iterated monodromy group and tuning are discussed.

(a) When is the IMG of a non-renormalizable quadratic polynomial of intermediate growth?

5. When are the IMGs of two polynomials f, g isomorphic?

A saturated isomorphism between weakly branch groups acting on a common tree is induced by an automorphism of the tree ([Nek05], Proposition 2.10.7). So if (i) both $\mathrm{IMG}(f)$ and $\mathrm{IMG}(g)$ are weakly branch, and (ii) every isomorphism $\phi : \mathrm{IMG}(f) \to \mathrm{IMG}(g)$ is saturated, then the limit (orbi)spaces associated to f and to g are isomorphic. For example, let $f(z) = z^2 + c_r$ and $g(z) = z^2 + c_a$ be the so-called rabbit and airplane polynomials, where $c_a \approx -1.7549$ and $c_r \approx -.1226 + .7449$ are respectively real and complex roots of $c^3 + 2c^2 + c + 1$. Nekrashevych ([Nek05], §3.11.2) shows that (i) and (ii) hold. Since the Julia sets are homeomorphic to the limit spaces, and the Julia sets are not isomorphic as orbispaces, one concludes that the IMGs of f and g are non-isomorphic as abstract groups. So it is natural to inquire

(a) When are the IMGs of polynomials weakly branch?

(b) When is an isomorphism $\phi : \mathrm{IMG}(f) \to \mathrm{IMG}(g)$ saturated?

A polynomial $f(z)$ is a *dynamical Belyi polynomial (DBP)* if $P_f = \{0, 1, \infty\}$. The coefficients of a DBP are necessarily algebraic. The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the set DBP by acting on the coefficients of polynomials. It is known that for any $n \in \mathbb{N}$, if f and g are Galois conjugate, then the monodromy group actions of f^n and g^n as coverings over $\mathbb{C} - \{0, 1\}$ are isomorphic.

If (i) and (ii) hold for Galois conjugate DBPs f, g, then $\mathrm{IMG}(f)$ and $\mathrm{IMG}(g)$ are isomorphic only when f and g are affine conjugate.

6. What are the dynamical and geometric implications of the amenability of IMGs for postcritically finite polynomials?

7. Let f be a postcritically finite polynomial. When is $\mathrm{IMG}(f)$ just infinite? A group is called *just-infinite* if every its proper quotient is finite. See [Gri00] for a relation between just-infinite groups and groups acting on rooted trees.

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