Rational Catalan combinatorics

The American Institute of Mathematics

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Corrections and new material are welcomed and can be sent to workshops@aimath.org

Version: Fri Dec 14 15:49:24 2012

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A.1 Armstrong, Drew

Apart from the explicitly stated goals of the workshop, I am interested more generally in learning about the algebra, combinatorics and geometry of V/G where G acts on V by reflections. What is the fundamental reason for extending this to $(V \oplus V^*)/G$? Since $V \oplus V^*$ is the cotangent bundle of V, it carries a symplectic structure, which allows us to think and do physics. But why are reflection groups particularly relevant? And what might be the geometric/physical significance of higher (e.g. tridiagonal) harmonics, whose algebraic and (more recently) combinatorial properties seem natural and interesting?

A.2 Balagovic, Martina

I am interested in Cherednik algebras, and in particular the problems relating to their representation theory (Verma modules, irreducible modules, category O). Recently I have been working with Prof Maxim Nazarov on trigonometric Cherednik algebras and its representation theory, using ideas from the representation theory of degenerate affine Hecke algebras, and links to combinatorics. I am hoping to learn more about the combinatorial side of things from this workshop, and find the interplay of combinatorics and representation theory extremely interesting.

A.3 Ballantine, Cristina

I am new to both the generalizations of Catalan numbers and Cherednik algebras and I hope the workshop will open new avenues for my research.

A.4 Bellamy, Gwyn

I am a representation theorist and interested in rational Cherednik algebras and their relationship to other areas of mathematics. In particular, I am interested in understanding how they can be used to talk problems in algebraic combinatorics. I am also interested in the relationship between these two areas and certain D-modules that appear naturally in Lie theory such as the Springer sheaf.

A.5 Bergeron, Francois

Both r-Dyck paths (or equivalent objects) and r-parking functions have been involved in interesting ways in setting up formulas for the character of the “bivariate coinvariant spaces” for the symmetric group. Many developments have eloquently confirmed the role of these combinatorial object in this context. Still the saga of the study of these spaces is far from over, and one of the main remaining open problem is to prove the the “Shuffle Conjecture” of [HHLRU], and its various refinements. This conjecture gives an entirely combinatorial description of the graded character for the bivariate case, all in terms of r-parking functions. Very recently, it has become apparent that the r-Tamari order is a main player in the extension of this saga to the case of the “tridiagonal coinvariant spaces”. In particular, it allows for an extension to the trivariate case of the Shuffle Conjecture, involving the enumeration of pairs of (α, φ), where φ is a r-parking function of “shape” β (a r-Dyck path), and α is smaller than β in r-Tamari order (see [trivariate]). Still, this extension has yet to be made more precise by the introduction the lacking third parameter.

Modulo some small adaptations, we may extend most of the above combinatorial considerations to the case $r = a/b$, for $a > b$ relatively prime integers. Enumeration formulas (for paths, as well as parking functions character) often remain nice. For example, when $r = 3/2$, the number of r-Dyck paths is given by the formula $\frac{1}{r\,n+1}\binom{(r+1)\,n}{n}$ whenever n is odd, and associated r-parking functions number $(r\,n+1)^{n-1}$; and this extends to enumerative properties of intervals for the r-Tamari lattice. Allong the lines of [armstrongreiner, rhoades], we would also like to extend, as far as possible, this nice interplay between combinatorics and algebra to other reflection (Coxeter) groups in the trivariate case; as well as to natural variants of the associahedron (Tamari lattice).

Bibliography

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[HHLRU] J. Haglund, M. Haiman, N. Loehr, J. Remmel, and A. Ulyanov, A Combinatorial Formula for the Character of the Diagonal Coinvariants, Duke Math. J. Volume 126, Number 2 (2005), 195-232.

[rhoades] B. Rhoades, Parking Structures: Fuss Analogs,
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A.6 Bezrukavniko, Roman

I am interested in algebraic geometry related to representations of rational Cherednik algebras and combinatorial questions needed to understand that geometry.

In particular, fixed points of a torus action on a Hilbert scheme of points on the plain (or more generally, on the resolution of the quotient of the plain by a cyclic group) are indexed by partitions (respectively, n-tuples of partitions). These fixed points are in bijection with representations of DAHA, either in zero or positive characteristic, for any value of the parameter of DAHA. Varying this parameter we get different representation categories related by wall-crossing. The sets of irreducible objects in categories related by a wall-crossing are in a natural bijection. Composing those bijections one gets an interesting (to me) permutation of the set of n-tuples of partitions, which in a particular case is related to the Mullineaux involution. I would like to better understand these bijections and some generalizations.

A.7 Garsia, Adriano

Basically there is an open problem closely related to Diagonal Harmonics. There seems to be a bi-graded module whose Frobenius characteristic is a multiple of Nabla nabla pn whose Frobenius characteristic perped by e1 gives $(n+1)^2$ multiple of the Frobenius characteristic of the Diagonal Harmonics.

The problem is to identify this module.

A.8 Gordon, Iain

I am interested in the representation theory of rational Cherednik algebras quite generally. I am particularly interested in understanding what sort of combinatorics in general appears when one tries to describe these representations, and to try to find geometric models for describing these representations. The Catalan combinatorics that appears is a prototypical examples of this.

A.9 Hicks, Angela

What Cherednik algebras can be used to show about the diagonal harmonics and diagonal harmonic alternants? Is it possible they offer a more elementary proof that the dimension of the diagonal harmonics (with n variables in both x and y) is $(n+1)^{n-1}$ or that the dimension of the diagonal harmonic alternants is counted by the Catalan numbers?

A.10 Lapointe, Luc

Let $\mathcal{A}_t^{(x)}$ be the t-antisymmetrization (or Hecke antisymmetrization) operator acting on the variables $x = x_1, \ldots, x_m$, let $\mathcal{S}_t^{(y)}$ be the t-symmetrization (or Hecke symmetrization) operator acting on the variables $y = y_1, \ldots, y_N$, let $\Delta_t(x)$ be the t-Vandermonde determinant in the variables x, and let $E_\gamma(x, y; q, t)$ be the non-symmetric Macdonald polynomials in the variables $x_1, \ldots, x_m, y_1, \ldots, y_N$. It is shown in arXiv:1211.3186 that if the first m parts of γ have no repeated parts and m and N are sufficiently large then we have (up to a multiplicative constant)

$$\frac{1}{\Delta_t(x)}\mathcal{A}_t^{(x)}\mathcal{S}_t^{(y)}E_\gamma(x, y; q, t) \propto P_\lambda^{(q,qt)}\left[X + \frac{q-qt}{1-qt}Y\right] P_\mu^{(qt,t)}[Y] =: P_{\lambda,\mu}(x, y; q, t)$$

where $\lambda+(m-1, \ldots, 0)$ is the rearrangement of the first m entries of γ, μ is the rearrangement of the last N entries of γ, and $P_\lambda^{(q,t)}[X]$ is the usual Macdonald polymomial.

The polynomials $P_{\lambda,\mu}(x, y; q, t)$ are thus natural bi-symmetric analog of the Macdonald polynomials. Remarkably, the Kostka and Nabla combinatorics of these new polynomials is that of the hyperoctahedral group B_n, that is, the corresponding Kostka coefficients $K_{\kappa,\nu\ \lambda,\mu}(q, t)$ are such that $K_{\kappa,\nu\ \lambda,\mu}(1, 1)$ is equal to the dimension of the irreducible representation of B_n indexed by the pairs of partitions (κ, ν). Moreover, it is possible to define a Nabla operator that produces a Frobenius series that should correspond to a natural B_n-module of dimension $(2n + 1)^n$.

I would be interested to know if the B_n-combinatorics of the polynomials $P_{\lambda,\mu}(x, y; q, t)$ has natural interpretations in the Cherednik algebra setting.

A.11 Loseu, Ivan

I am a representation theorist and is interested, in particular, in Rational Cherednik algebras. I hope to learn more about their connections to Combinatorics.

A.12 Mazin, Mikhail

I am interested in the generalized q, t-Catalan numbers in connection with topological invariants of quasihomogeneous plane curve singularities, as well as the geometry of the Hilbert scheme of points on the complex plane. The generalized q, t-Catalan numbers are defined by analogy with Haiman's combinatorial formula for the classical q, t-Catalan number:

$$c_{m,n}(q, t) = \sum_D q^{\delta-|D|}t^{h^{\frac{n}{m}}(D)},$$

where the sum is taken over all Young diagrams inside a right $n \times m$ triangle. The statistic $h^{\frac{n}{m}}$, originally defined in connection with the study of cell decompositions of the Hilbert scheme of points on the plane, is given by the following formula:

$$h^x(D) = \left| \left\{ c \in D : \frac{a(c)}{l(c)+1} < x < \frac{a(c)+1}{l(c)} \right\} \right|,$$

where $a(c)$ and $l(c)$ are the *arm* and the *leg* of the box $c \in D$ correspondingly. In the case $m = n+1$ one gets $h^{\frac{n}{n+1}}(D) = dinv(D)$, and $c_{m,n}(q,t) = c_n(q,t)$ by Haiman's combinatorial formula. The following two conjectures remain widely open beyond just a handful of cases:

Symmetry conjecture The function $c_{m,n}(q,t)$ satisfies the functional equation

$$c_{m,n}(q,t) = c_{m,n}(t,q).$$

Weak symmetry conjecture The function $c_{m,n}(q,t)$ satisfies the functional equation

$$c_{m,n}(q,1) = c_{m,n}(1,q).$$

The first conjecture is known for the classical q,t-Catalan numbers ($m = n+1$ case), and for $n \leq 4$. The second is known for $m = kn \pm 1$.

There is a number of, mostly conjectural, connections to different fields:

A. Finite dimensional representations of Cherednik algebras;
B. Homological knot invariants of torus knots;
C. Hilbert polynomials of spaces of global sections of certain sheaves on the Hilbert scheme (Haiman-type description for the generalized case);
D. Perverse filtrations on the cohomology of Jacobi factors of quasihomogeneous plane curve singularities.

I am looking forward to participating in the workshop and learning more about the above connections from the leading specialists in the corresponding areas. This seems to be a fast emerging area, with a lot of exciting open problems.

A.13 Niese, Elizabeth

My research has primarily been in the area of combinatorial Macdonald polynomials. More recently I have been looking for an injection from the standard fillings that generate the Hilbert series of the Garsia-Haiman module and the parking functions that generate the conjectured combinatorial definition of the Hilbert series of the diagonal harmonics. With Nick Loehr I have also been looking at alternate versions of the "shuffle conjecture" using the bounce and area' statistics of Haglund and Loehr on labeled Dyck paths and the area, pmaj statistics of Loehr, as well as alternate versions of the higher order shuffle conjecture.

As I am still very early in my career, I am looking for ways to expand my research program. With the connections between labeled Dyck paths in rectangles and rational Catalan numbers, this workshop is a natural way for me to learn and contribute to an emerging area of algebraic combinatorics.

A.14 Reading, Nathan

I am interested in learning more about the double affine Hecke algebra and Cherednik algebras and their relationship with sortable elements, generalized associahedra and non-crossing partitions.

A.15 Reiner, Victor

I am a combinatorialist, interested in understanding more about how rational Cherednik algebras can help us to resolve one of the big remaining mysteries in Catalan combinatorics:

Why does the W-Catalan number

$$\prod_{i=1}^{n} \frac{h+d_i}{d_i}?$$

where W is a real reflection group with fundamental degrees $(d_1, \ldots, d_n)$ and Coxeter number h, give the cardinality of the set

$$NC(W) := \{w \in W : \ell_T(w) + \ell_T(w^{-1}c) = n\}$$

of W-noncrossing partitions, where c any choice of a Coxeter element, and $\ell_T(-)$ the absolute length function, that is, the length with respect to the generating set of all reflections T.

The arXiv preprint by D. Armstrong, B. Rhoades and myself titled "Parking spaces" gives a main conjecture which is my current hope for finding such a connection to the RCA theory.

A.16 Rhoades, Brendon

My starting point in studying Catalan combinatorics was in an effort to generalize the combinatorial aspects of parking functions to a broader class of reflection groups. Given any real reflection group W, in joint work with D. Armstrong and V. Reiner I defined a set of 'noncrossing parking functions' Park_W^{NC} which generalize the combinatorial notion of a parking function in type A. The set ${}_W^{NC}$ carries an action of $W \times C$, where C is the cyclic group generated by a Coxeter element $c \in W$. In further solo work, I defined a combinatorial Fuss analog $\mathsf{Park}_W^{NC}(k)$ depending on a Fuss parameter $k \geq 1$; this structure carries an action of $W \times \mathbb{Z}_{kh}$, where h is the Coxeter number of W. To determine the characters of these $W \times \mathbb{Z}_{kh}$-sets, we analyze a construction called the 'algebraic parking space' whose existence comes from the theory of rational Cherednik algebras and whose character can be computed using algebraic techniques. Geometrically, the algebraic parking space can be interpreted as an action of $W \times \mathbb{Z}_{kh}$ on a certain zero-dimensional subscheme of the reflection representation V.

When W is a well-generated complex reflection group, we can define a 'noncrossing parking space' and its Fuss analog in a similar way. How much of the theory of rational Cherednik algebras carries through to this broader setting? In particular, does there always exist a homogenous system of parameters in the coordinate ring $\mathbb{C}[V]$ of appropriate degree carrying (a Galois twist of?) V^* whose corresponding ideal we can deform to yield an algebraic parking space? To what extent can this theory be pushed to even broader classes of groups W? (For example, can we drop the well generation hypothesis?)

In a different direction, for any real reflection group W with Coxeter number h and any $p > h$ such that p and h are coprime, general rational Cherednik algebra theory guarantees the existence of a h.s.o.p. of degree p inside $\mathbb{C}[V]$ carrying a Galois twist of V^*. This allows us to define an 'algebraic parking space' which is a zero-dimensional reduced subscheme of V and carries an action of $W \times \mathbb{Z}_{p-1}$. However, except in the case where $p \equiv 1 \pmod{h}$, there is no known *combinatorial* model for the parking space analogous to Park_W^{NC}, even when $W = S_n$ is of type A.

For any $p > h$ such that p and h are coprime, build a combinatorial model for $\mathsf{Park}_W^{NC}(p)$ which carries an action of $W \times \mathbb{Z}_{p-1}$ and is isomorphic as a $W \times \mathbb{Z}_{p-1}$-module to the corresponding 'algebraic' parking space.

In type A, a candidate for such a model comes from the new theory of 'rational Catalan combinatorics' of D. Armstrong. If $b > a$ are coprime, one can consider a natural 'a, b-analog' of Dyck paths to define a set of 'a, b-noncrossing partitions' which embed inside the lattice of noncrossing partitions of $[b-1]$. The a, b-noncrossing partitions can be decorated in a natural way to give rise to objects which may deserve to be called a, b-parking functions and which may carry a nice action of $S_a \times \mathbb{Z}_{b-1}$. However, this lattice path construction leaves it unclear that these a, b-noncrossing partitions are even closed under rotation. Getting a better handle on the type A case of Problem would seem to help in a broader attack.

A.17 Ripoll, Vivien

I am interested in the combinatorics and geometry of Coxeter groups and complex reflection groups, and I would be happy to learn more about Cherednik algebras and how they can help tackle combinatorial problems of this area.

More precisely, I am interested in the combinatorics of the generalized noncrossing partition lattice $NC(W)$, associated to any finite Coxeter group, or even complex reflection group W. The cardinality of (W) (for W irreducible) is given by a uniform formula involving the invariant degrees $(d_1 \leq d_2 \leq \cdots \leq d_n = h)$ of W:

$$\prod_{i=1}^{n} \frac{d_i + h}{d_i} \quad (1),$$

which is called the W-Catalan number $Cat(W)$ (in type A this is the usual Catalan number). More generally, the number of multichains of a given length (say length p) in $NC(W)$ is equal to

$$\prod_{i=1}^{n} \frac{d_i + ph}{d_i} \quad (2),$$

called the W-Fuss-Catalan number $Cat^{(p)}(W)$.

To this date, as far as I know, there is still no uniform proof of these equalities (even for Formula (1)), only case-by-case proofs using the classification of irreducible reflection groups. A uniform comprehension of these kind of questions could come from geometric methods.

In 2006 David Bessis opened an avenue towards a geometric interpretation of these formulas. First, note that the numerology of strict chains is easily related to the one of multichains; and these strict chains are in fact in bijection with some "minimal" factorizations of a Coxeter element c of W in a certain number of factors. For example, Formula (2) implies that the number of factorizations of c in n reflections (= the number of maximal strict chains in $NC(W)$) equals $n!h/|W|$ (this equality alone has also no case-free proof). It turns out that this number is also the degree of some ramified covering constructed from the geometry of W, called the Lyasko-Looijenga covering LL. Bessis used this fact to relate the maximal factorizations of c with the generic fibers of LL. He constructed these factorizations geometrically, using some decompositions of loops in the braid group of W. During my thesis, I generalized these results to some more general factorizations of c, and I deduced from geometric properties of LL some combinatorial results for the factorisations (and thus the multichains), using only case-free constructions. I was unable to prove uniformly the Fuss-Catalan formula, but I became convinced (and I still am) that this formula should come from a detailed algebraic understanding of the fibers of the covering LL. I would like to understand if the framework of Cherednik algebras can help studying these objects.

A.18 Sommers, Eric

I'm interested in the connections between representation theory and the combinatorics that shows up in various q- or (q,t)- analogues of the Catalan numbers. I'm particularly fascinated with the connections to ordinary Springer theory and to affine Springer theory, especially when one ventures outside of type A.

A.19 Thomas, Hugh

I am interested in Catalan phenomena related to cluster algebras and/or noncrossing partitions, and their manifestation in representation theory of (mostly hereditary) algebras. See in particular [arXiv:math/0612219] with Colin Ingalls, [arXiv:1007.0928] with Idun Reiten and Aslak Bakke Buan, and [arXiv:1203.0277] with David Speyer.

I am also interested in the connection (or lack of connection) between noncrossing and nonnesting partitions. See in particular [arXiv:1101.1277] with Drew Armstrong and Christian Stump.

An idea which seems to me to fit in the "rational Catalan" framework occurred to me recently. At the moment, it is mostly a way to dress up some rather simple combinatorics in some fancy language.

Consider a quiver with two vertices, which we denote X and Y, with arrows from X to Y and from Y to X, and loops at both X and Y. Let us consider uniserial representations of this quiver. Such representations correspond bijectively to monomials in non-commuting variables x and y. We can also draw pictures of these representations as lattice paths from $(0, 0)$ to (a, b), where a is the number of occurrences of x in the monomial, and b is the number of occurrences of y in the monomial: an x is encoded by a right step, and a y by an up step.

Now, if we're doing rational Catalan combinatorics, we are interested in paths from $(0, 0)$ to (a, b) which lie strictly above the straight line from $(0, 0)$ to (a, b) except at the endpoints. Slightly abusing terminology, call such paths Dyck paths. It turns out that, among the representations with a many X's and b many Y's, these are exactly the *stable* representations, in the sense of Bridgeland [arXiv:math/0212237], with respect to a suitable choice of stability function.

Some algebraic notions have simple combinatorial manifestations in this setting. The Harder-Narasimhan filtration is nothing but the observation that any finite-length path starting at $(0, 0)$ and proceeding up and to the right, can be factored into Dyck paths of weakly increasing slopes. Expressing this as a generating function, we get that the generating function for paths, i.e. $1/1-(x+y)$, is equal to a product over all rational slopes, in weakly increasing order, of a generating function for a product of Dyck paths of that slope. This is a rather simple version of Kontsevich-Soibelmann wall-crossing, since we can think of the lefthand side as arising from a different choice of stability function. (See the first two sections of Keller's [arXiv:1102.4148] for an introduction to the representation-theoretic notions at work here.) Note also that by setting $yx = qxy$, which is also natural from the wall-crossing point of view, we get a generating function for paths with respect to the area below them.