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The topology of $Out(F_n)$

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1. Introduction

The aim of this note is to survey some of the topological methods developed in the last 20 years to study the group $Out(F_n)$ of outer automorphisms of a free group F_n of rank n. For an excellent and more detailed survey see also [69]. Stallings' paper [64] marks the turning point and for the earlier history of the subject the reader is referred to [55]. $Out(F_n)$ is defined as the quotient of the group $Aut(F_n)$ of all automorphisms of F_n by the subgroup of inner automorphisms. On one hand, abelianizing F_n produces an epimorphism $Out(F_n) \to Out(\mathbb{Z}^n) = GL_n(\mathbb{Z})$, and on the other hand $Out(F_n)$ contains as a subgroup the mapping class group of any compact surface with fundamental group F_n. A *leitmotiv* in the subject, promoted by Karen Vogtmann, is that $Out(F_n)$ satisfies a mix of properties, some inherited from mapping class groups, and others from arithmetic groups. The table below summarizes the parallels between topological objects associated with these groups.

Mapping class groups	$Out(F_n)$	$GL_n(\mathbb{Z})$ **(arithmetic groups)**	**algebraic properties**
Teichmüller space	Culler-Vogtmann's Outer space	$GL_n(R)/O_n$ (symmetric spaces)	finiteness properties cohomological dimension
Thurston normal form	train track representative	Jordan normal form	growth rates fixed points (subgroups)
Harer's bordification	bordification of Outer space	Borel-Serre bordification	Bieri-Eckmann duality
measured laminations	**R**-trees	flag manifold (Furstenberg boundary)	Kolchin theorem Tits alternative
Harvey's curve complex	?	Tits building	rigidity

Outer space is not a manifold and only a polyhedron, imposing a combinatorial character on $Out(F_n)$.

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2. Stallings' Folds

A *graph* is a 1-dimensional cell complex. A map $f : G \to G'$ between graphs is *simplicial* if it maps vertices to vertices and open 1-cells homeomorphically to open 1-cells. The simplicial map f is a *fold* if it is surjective and identifies two edges that share at least one vertex. A fold is a homotopy equivalence unless the two edges share both pairs of endpoints and in that case the induced homomorphism in π_1 corresponds to killing a basis element.

Theorem 1 (Stallings [63]). *A simplicial map $f : G \to G'$ between finite connected graphs can be factored as the composition*

$$G = G_0 \to G_1 \to \cdots \to G_k \to G'$$

where each $G_i \to G_{i+1}$ is a fold and $G_k \to G'$ is locally injective (an immersion). Moreover, such a factorization can be found by a (fast) algorithm.

In the absence of valence 1 vertices the last map $G_k \to G'$ can be thought of as the core of the covering space of G' corresponding to the image in π_1 of f. The following problems can be solved algorithmically using Theorem 1 (these were known earlier, but Theorem 1 provides a simple unified argument). Let F be a free group with a fixed finite basis.

- Find a basis of the subgroup H generated by a given finite collection $h_1, \cdots, h_k$ of elements of F.
- Given $w \in F$, decide if $w \in < h_1, \cdots, h_k >$.
- Given $w \in F$, decide if w is conjugate into $< h_1, \cdots, h_k >$.
- Given a homomorphism $\phi : F \to F'$ between two free groups of finite rank, decide if ϕ is injective, surjective.
- Given finitely generated $H < F$ decide if it has finite index.
- Given two f.g. subgroups $H_1, H_2 < F$ compute $H_1 \cap H_2$ and also the collection of subgroups $H_1 \cap H_2^g$ where $g \in F$. In particular, is H_1 malnormal?
- Represent a given automorphism of F as the composition of generators of $Aut(F)$ of the following form:
  Signed permutations: each a_i maps to a_i or to a_i^{-1}.
  Change of maximal tree: $a_1 \mapsto a_1$ and for $i > 1$ a_i maps to one of $a_1^{\pm 1} a_i$ or to $a_i a_1^{\pm 1}$.
- Todd-Coxeter process [65].

3. Culler-Vogtmann's Outer space

Fix the wedge of n circles R_n and a natural identification $\pi_1(R_n) \cong F_n$ in which oriented edges correspond to the basis elements. Thus any $\phi \in Out(F_n)$ can be thought of as a homotopy equivalence $R_n \to R_n$. A *marked metric graph* is a pair (G, g) where

- G is a finite graph without vertices of valence 1 or 2.
- $g : R_n \to G$ is a homotopy equivalence (the *marking*).

- G is equipped with a path metric so that the sum of the lengths of all edges is 1.

Outer space X_n is the set of equivalence classes of marked metric graphs under the equivalence relation $(G, g) \sim (G', g')$ if there is an isometry $h : G \to G'$ such that gh and g' are homotopic [28].

If α is a loop in R_n we have the length function $l_\alpha : X_n \to \mathbb{R}$ where $l_\alpha(G, g)$ is the length of the immersed loop homotopic to $g(\alpha)$. The collection $\{l_\alpha\}$ as α ranges over all immersed loops in R_n defines an injection $X_n \to \mathbb{R}^\infty$ and the topology on X_n is defined so that this injection is an embedding. X_n naturally decomposes into open simplices obtained by varying edge-lengths on a fixed marked graph. The group $Out(F_n)$ acts on X_n on the right via

$$(G, g)\phi = (G, g\phi)$$

Theorem 2 (Culler-Vogtmann [28]). *X_n is contractible and the action of $Out(F_n)$ is properly discontinuous (with finite point stabilizers). X_n equivariantly deformation retracts to a $(2n-3)$-dimensional complex.*

If (G, g) and (G', g') represent two points of X_n, there is a "difference of markings" map $h : G \to G'$ such that hg and g' are homotopic. Representing h as a composition of folds (appropriately interpreted) leads to a path in X_n from (G, g) to (G', g'). Arranging that these paths vary continuously with endpoints leads to a proof of contractibility of X_n [66],[60],[71].

Corollary 3. *The virtual cohomological dimension* $\mathrm{vcd}(Out(F_n)) = 2n - 3$.

Theorem 4 (Culler [26]). *Every finite subgroup of $Out(F_n)$ fixes a point of X_n.*

Outer space can be equivariantly compactified [27]. Points at infinity are represented by actions of F_n on $\mathbb{R}$-trees.

4. Train tracks

Any $\phi \in Out(F_n)$ can be represented as a cellular map $f : G \to G$ on a marked graph G. We say that ϕ is *reducible* if there is such a representative where

- G has no vertices of valence 1 or 2, and
- there is a proper f-invariant subgraph of G with at least one non-contractible component.

Otherwise, we say that ϕ is *irreducible*.

A cellular map $f : G \to G$ is a *train track map* if for every $k > 0$ the map $f^k : G \to G$ is locally injective on every open 1-cell. For example, homeomorphisms are train track maps and Culler's theorem guarantees that every $\phi \in Out(F_n)$ of finite order has a representative $f : G \to G$ which is a homeomorphism. More generally, we have

Theorem 5 (Bestvina-Handel [12]). *Every irreducible outer automorphism ϕ can be represented as a train track map $f : G \to G$.*

Any vertex $v \in G$ has a cone neighborhood, and the frontier points can be thought of as "germs of edges" at v. A train track map (or any cellular map that does not collapse edges) f induces the "derivative" map Df on these germs (on possibly different vertices). We declare two germs at the same vertex to be equivalent (and the corresponding "turn" *illegal*) if they get identified by some power of Df (and otherwise the turn is *legal*). An immersed loop in G is *legal* if every turn determined by entering and then exiting a vertex is legal. It follows that f sends legal loops to legal loops. This gives a method for computing the growth rate of ϕ, as follows. The *transition matrix* (a_{ij}) of f (or more generally of a cellular map $G \to G$ that is locally injective on edges) has a_{ij} equal to the number of times that the f-image of j^{th} edge crosses i^{th} edge. Applying the Perron-Frobenius theorem to the transition matrix, one can find a unique metric structure on G such that f expands lengths of edges (and also legal loops) by a factor $\lambda \geq 1$. For a conjugacy class γ in F_n the growth rate is defined as

$$GR(\phi, \gamma) = \limsup_{k \to \infty} \log(||\phi^k(\gamma)||)/k$$

where $||\gamma||$ is the word length of the cyclically reduced word representing γ. Growth rates can be computed using lengths of loops in G rather than in R_n.

Corollary 6. *If ϕ is irreducible as above, then either γ is a ϕ-periodic conjugacy class, or $GR(\phi, \gamma) = \log \lambda$. Moreover,* $\limsup$ *can be replaced by* $\lim$.

The proof of Theorem 5 uses a folding process that successively reduces the Perron-Frobenius number of the transition matrix until either a train track representative is found, or else a reduction of ϕ is discovered. This process is algorithmic (see [13],[21]).

Another application of train tracks is to fixed subgroups.

Theorem 7 (Bestvina-Handel [12]). *Let $\Phi : F_n \to F_n$ be an automorphism whose associated outer automorphism is irreducible. Then the fixed subgroup $Fix(\Phi)$ is trivial or cyclic. Without the irreducibility assumption, the rank of $Fix(\Phi)$ is at most n.*

It was known earlier by the work of Gersten [39] that $Fix(\Phi)$ has finite rank (for simpler proofs see [42],[25]). The last sentence in the above theorem was conjectured by Peter Scott. Subsequent work by Collins-Turner [24], Dicks-Ventura [31], Ventura [68], Martino-Ventura [58], imposed further restrictions on a subgroup of F_n that occurs as the fixed subgroup of an automorphism. To analyze reducible automorphisms, a more general version of a train track map is required.

Definition 8. A cellular map $f : G \to G$ on a finite graph with no vertices of valence 1 that does not collapse any edges is a *relative train track map* if there is a filtration

$$\emptyset = G_0 \subset \cdots \subset G_m = G$$

into f-invariant subgraphs with the following properties. Denote by H_r the closure of $G_r \backslash G_{r-1}$, and by M_r the part of the transition matrix corresponding to H_r. Then M_r is the zero matrix or an irreducible matrix. If M_r is irreducible and the Perron-Frobenius eigenvalue $\lambda_r > 1$ then:

- the derivative Df maps the germs in H_r to germs in H_r,
- if α is a nontrivial path in G_{r-1} with endpoints in $G_{r-1} \cap H_r$ then $f(\alpha)$, after pulling tight, is also a nontrivial path with endpoints in $G_{r-1} \cap H_r$, and
- every legal path in H_r is mapped to a path that does not cross illegal turns in H_r.

As an example, consider the automorphism $a \mapsto a, b \mapsto ab, c \mapsto caba^{-1}b^{-1}d$, $d \mapsto dbcd$ represented on the rose R_4. The strata are $\emptyset \subset G_1 = \{a\} \subset \{a, b\} \subset G$. H_1 and H_2 have $\lambda = 1$ while H_3 has $\lambda_3 > 1$. The following is an analog of Thurston's normal form for surface homeomorphisms.

Theorem 9. [12] *Every automorphism of F_n admits a relative train track representative.*

Consequently, automorphisms of F_n can be thought of as being built from building blocks (exponential and non-exponential kinds) but the later stages are allowed to map over the previous stages. This makes the study of automorphisms of F_n more difficult (and interesting) than the study of surface homeomorphisms. Other non-surface phenomena (present in linear groups) are:

- stacking up non-exponential strata produces (nonlinear) polynomial growth,
- the growth rate of an automorphism is generally different from the growth rate of its inverse.

5. Related spaces and structures

Unfortunately, relative train track representatives are far from unique. As a replacement, one looks for canonical objects associated to automorphisms that can be computed using relative train tracks. There are 3 kinds of such objects, all stemming from the surface theory: laminations, $\mathbb{R}$-trees, and hierarchical decompositions of F_n [59].
Laminations. Laminations were used in the proof of the Tits alternative for $Out(F_n)$. To each automorphism one associates finitely many attracting laminations. Each consists of a collection of "leaves", i.e. biinfinite paths in the graph G, or alternatively, of an F_n-orbit of pairs of distinct points in the Cantor set of ends of F_n. A leaf ℓ can be computed by iterating an edge in an exponentially growing stratum H_r. The other leaves are biinfinite paths whose finite subpaths appear as subpaths of ℓ. Some of the attracting laminations may be sublaminations of other attracting laminations, and one focuses on the maximal (or *topmost*) laminations. It is possible to identify the basin of attraction for each such lamination. Let $\mathcal{H}$ be any subgroup of $Out(F_n)$. Some of the time it is possible to find to elements $f, g \in \mathcal{H}$ that attract each other's laminations and then the standard ping-pong argument shows that $< f, g >\cong F_2$. Otherwise, there is a finite set of attracting laminations permuted by $\mathcal{H}$, a finite index subgroup $\mathcal{H}_0 \subset \mathcal{H}$ that fixes each of these laminations and a homomorphism ("stretch factor") $\mathcal{H}_0 \to A$ to a finitely generated abelian group A whose kernel consists entirely of polynomially growing automorphisms. There is an analog of Kolchin's theorem that says that finitely generated groups of polynomially growing automorphisms can simultaneously be realized as relative train track maps on the same graph (the classical Kolchin theorem

says that a group of unipotent matrices can be conjugated to be upper triangular, or equivalently that it fixes a point in the flag manifold). The main step in the proof of the analog of Kolchin's theorem is to find an appropriate fixed $\mathbb{R}$-tree in the boundary of Outer space. This leads to the Tits alternative for $Out(F_n)$:

Theorem 10 (Bestvina-Feighn-Handel [9],[10],[7]). *Any subgroup $\mathcal{H}$ of $Out(F_n)$ either contains F_2 or is virtually solvable.*

A companion theorem [8] (for a simpler proof see [1]) is that solvable subgroups of $Out(F_n)$ are virtually abelian.

$\mathbb{R}$-trees. Points in the compactified Outer space are represented as F_n-actions on $\mathbb{R}$-trees. It is then not surprising that the Rips machine [5], which is used to understand individual actions, provides a new tool to be deployed to study $Out(F_n)$. Gaboriau, Levitt, and Lustig [37] and Sela [59] find another proof of Theorem 7. Gaboriau and Levitt compute the topological dimension of the boundary of Outer Space [36]. Levitt and Lustig show [51] that automorphisms with irreducible powers have the standard north-south dynamics on the compactified Outer space. Guirardel [43] shows that the action of $Out(F_n)$ on the boundary does not have dense orbits; however, there is a unique minimal closed invariant set. For other applications of $\mathbb{R}$-trees in geometric group theory, the reader is referred to the survey [2].

Cerf theory. An advantage of $Aut(F_n)$ over $Out(F_n)$ is that there is a natural inclusion $Aut(F_n) \to Aut(F_{n+1})$. One can define *Auter Space* AX_n similarly to Outer space, except that all graphs are equipped with a base vertex, which is allowed to have valence 2. The degree of the base vertex v is $2n - \text{valence}(v)$. Denote by D_n^k the subcomplex of AX_n consisting of graphs of degree $\leq k$. Hatcher-Vogtmann [47] develop a version of Cerf theory and show that D_n^k is $(k-1)$-connected. Since the quotient $D_n^k/Aut(F_n)$ stabilizes when n is large, one sees that (rational) homology $H_i(Aut(F_n))$ also stabilizes when n is large ($n \geq 3i/2$). Hatcher-Vogtmann show that the same is true for integral homology and in the range $n \geq 2i+3$. They also make explicit computations in low dimensions [49] and all stable rational homology groups H_i vanish for $i \leq 7$.

Bordification. The action of $Out(F_n)$ on Outer space X_n is not cocompact. By analogy with Borel-Serre bordification of symmetric spaces [14] and Harer's bordification of Teichmüller space [44], Bestvina and Feighn [6] bordify X_n, i.e. equivariantly add ideal points so that the action on the new space BX_n is cocompact. This is done by separately compactifying every simplex with missing faces in X_n and then gluing these together. To see the idea, consider the case of the theta-graph in rank 2. Varying metrics yields a 2-simplex σ without the vertices. As a sequence of metrics approaches a missing vertex, the lengths of two edges converge to 0. Restricting a metric to these two edges and normalizing so that the total length is 1 gives a point in $[0,1]$ (the length of one of the edges), and a way to compactify σ by adding an interval for each missing vertex. The compactified σ is a hexagon. This procedure equips the limiting theta graph with a metric that may vanish on two edges, in which case a "secondary metric" is defined on their union. In general, a graph representing a point in the bordification is equipped with a sequence of metrics, each defined on the core of the subgraph where the previous metric vanishes.

Lengths of curves (at various scales) provide a "Morse function" on BX_n with values in a product of $[0,\infty)$'s with the target lexicographically ordered. The sublevel and superlevel sets intersect each cell in a semi-algebraic set and it is possible to study how the homotopy types change as the level changes. A distinct advantage of BX_n over the spine of X_n (an equivariant deformation retract) is that the change in homotopy type of superlevel sets as the level decreases is very simple – via attaching of cells of a fixed dimension.

Theorem 11 (Bestvina-Feighn [6]). *BX_n and $Out(F_n)$ are $(2n-5)$-connected at infinity, and $Out(F_n)$ is a virtual duality group of dimension $2n-3$.*

Mapping tori. If $\phi : F_n \to F_n$ is an automorphism, form the mapping torus $M(\phi)$. This is the fundamental group of the mapping torus $G \times [0,1]/(x,1) \sim (f(x),0)$ of any representative $f : G \to G$, and it plays the role analogous to 3-manifolds that fiber over the circle. Such a group is always coherent [33]. A quasi-isometry classification of these groups seems out of reach, but the following is known. When ϕ has no periodic conjugacy classes, $M(\phi)$ is a hyperbolic group [20]. When ϕ has polynomial growth, $M(\phi)$ satisfies quadratic isoperimetric inequality [57] and moreover, $M(\phi)$ quasi-isometric to $M(\psi)$ for ψ growing polynomially forces ψ to grow as a polynomial of the same degree [56]. Bridson and Groves announced [16] that $M(\phi)$ satisfies quadratic isoperimetric inequality for all ϕ.

Geometry. Perhaps the biggest challenge in the field is to find a good geometry that goes with $Out(F_n)$. The payoff would most likely include rigidity theorems for $Out(F_n)$. Both mapping class groups and arithmetic groups act isometrically on spaces of nonpositive curvature. Unfortunately, the results to date for $Out(F_n)$ are negative. Bridson [15] showed that Outer space does not admit an equivariant piecewise Euclidean $CAT(0)$ metric. $Out(F_n)$ $(n > 2)$ is far from being $CAT(0)$ [17],[40].

An example of a likely rigidity theorem is that higher rank lattices in simple Lie groups do not embed into $Out(F_n)$. A possible strategy is to follow the proof in [11] of the analogous fact for mapping class groups. The major missing piece of the puzzle is the replacement for Harvey's curve complex; a possible candidate is described in [48].

References

[1] Emina Alibegović, *Translation lengths in $Out(F_n)$*, 2000, to appear.

[2] Mladen Bestvina, **R**-*trees in topology, geometry, and group theory*, Handbook of geometric topology (R.J. Daverman and R.B. Sher, eds.), Elsevier Science B.V., 2002.

[3] Mladen Bestvina and Noel Brady, *Morse theory and finiteness properties of groups*, Invent. Math. **129** (1997), no. 3, 445–470.

[4] Mladen Bestvina and Mark Feighn, *A combination theorem for negatively curved groups*, J. Differential Geom. **35** (1992), no. 1, 85–101, Addendum and correction: **43** (1996), no. 4, 783–788.

[5] ———, *Stable actions of groups on real trees*, Invent. Math. **121** (1995), no. 2, 287–321.

[6] ———, *The topology at infinity of* $\mathrm{Out}(F_n)$, Invent. Math. **140** (2000), no. 3, 651–692.

[7] Mladen Bestvina, Mark Feighn, and Michael Handel, *The Tits Alternative for Out(F_n) II: a Kolchin Theorem*, to appear, 1996.
[8] ______, *The Tits Alternative for Out(F_n) III: Solvable Subgroups*, preprint, 1996.
[9] ______, *Laminations, trees, and irreducible automorphisms of free groups*, Geom. Funct. Anal. **7** (1997), no. 2, 215–244, Erratum: **7** (1997), no. 6, 1143.
[10] ______, *The Tits alternative for* $\mathrm{Out}(F_n)$. *I. Dynamics of exponentially-growing automorphisms*, Ann. of Math. (2) **151** (2000), no. 2, 517–623. MR 2002a:20034
[11] Mladen Bestvina and Koji Fujiwara, *Bounded cohomology of subgroups of mapping class groups*, Geom. Top. **6** (2002), 69–89.
[12] Mladen Bestvina and Michael Handel, *Train tracks and automorphisms of free groups*, Ann. of Math. (2) **135** (1992), no. 1, 1–51.
[13] ______, *Train-tracks for surface homeomorphisms*, Topology **34** (1995), no. 1, 109–140.
[14] A. Borel and J.-P. Serre, *Corners and arithmetic groups*, Comment. Math. Helv. **48** (1973), 436–491, Avec un appendice: Arrondissement des variétés à coins, par A. Douady et L. Hérault.
[15] Martin R. Bridson, *Geodesics and curvature in metric simplicial complexes*, Group theory from a geometrical viewpoint (Trieste, 1990), World Sci. Publishing, River Edge, NJ, 1991, pp. 373–463.
[16] Martin R. Bridson and Daniel Groves, *The quadratic isoperimetric inequality for mapping tori of free group automorphisms I: Positive automorphisms*, preprint.
[17] Martin R. Bridson and Karen Vogtmann, *On the geometry of the automorphism group of a free group*, Bull. London Math. Soc. **27** (1995), no. 6, 544–552.
[18] ______, *Automorphisms of automorphism groups of free groups*, J. Algebra **229** (2000), no. 2, 785–792.
[19] ______, *The symmetries of Outer space*, Duke Math. J. **106** (2001), no. 2, 391–409.
[20] Peter Brinkmann, *Hyperbolic automorphisms of free groups*, Geom. Funct. Anal. **10** (2000), no. 5, 1071–1089.
[21] ______, *An implementation of the Bestvina-Handel algorithm for surface homeomorphisms*, Experiment. Math. **9** (2000), no. 2, 235–240.
[22] Marshall M. Cohen and Martin Lustig, *On the dynamics and the fixed subgroup of a free group automorphism*, Invent. Math. **96** (1989), no. 3, 613–638.
[23] ______, *Very small group actions on* $\mathbf{R}$*-trees and Dehn twist automorphisms*, Topology **34** (1995), no. 3, 575–617.
[24] D. J. Collins and E. C. Turner, *All automorphisms of free groups with maximal rank fixed subgroups*, Math. Proc. Cambridge Philos. Soc. **119** (1996), no. 4, 615–630.
[25] Daryl Cooper, *Automorphisms of free groups have finitely generated fixed point sets*, J. Algebra **111** (1987), no. 2, 453–456.
[26] Marc Culler, *Finite groups of outer automorphisms of a free group*, Contributions to group theory, Amer. Math. Soc., Providence, RI, 1984, pp. 197–207.
[27] Marc Culler and John W. Morgan, *Group actions on* $\mathbf{R}$*-trees*, Proc. London Math. Soc. (3) **55** (1987), no. 3, 571–604.
[28] Marc Culler and Karen Vogtmann, *Moduli of graphs and automorphisms of free groups*, Invent. Math. **84** (1986), no. 1, 91–119.

[29] ———, *The boundary of Outer space in rank two*, Arboreal group theory (Berkeley, CA, 1988), Springer, New York, 1991, pp. 189–230.
[30] ———, *A group-theoretic criterion for property* FA, Proc. Amer. Math. Soc. **124** (1996), no. 3, 677–683.
[31] Warren Dicks and Enric Ventura, *Irreducible automorphisms of growth rate one*, J. Pure Appl. Algebra **88** (1993), no. 1-3, 51–62.
[32] ———, *The group fixed by a family of injective endomorphisms of a free group*, American Mathematical Society, Providence, RI, 1996.
[33] Mark Feighn and Michael Handel, *Mapping tori of free group automorphisms are coherent*, Ann. of Math. (2) **149** (1999), no. 3, 1061–1077.
[34] Edward Formanek and Claudio Procesi, *The automorphism group of a free group is not linear*, J. Algebra **149** (1992), no. 2, 494–499.
[35] Damien Gaboriau, Andre Jaeger, Gilbert Levitt, and Martin Lustig, *An index for counting fixed points of automorphisms of free groups*, Duke Math. J. **93** (1998), no. 3, 425–452.
[36] Damien Gaboriau and Gilbert Levitt, *The rank of actions on* **r***-trees*, Ann. Sci. École Norm. Sup. (4) **28** (1995), no. 5, 549–570.
[37] Damien Gaboriau, Gilbert Levitt, and Martin Lustig, *A dendrological proof of the Scott conjecture for automorphisms of free groups*, Proc. Edinburgh Math. Soc. (2) **41** (1998), no. 2, 325–332.
[38] S. M. Gersten, *A presentation for the special automorphism group of a free group*, J. Pure Appl. Algebra **33** (1984), no. 3, 269–279.
[39] ———, *Fixed points of automorphisms of free groups*, Adv. in Math. **64** (1987), no. 1, 51–85.
[40] ———, *The automorphism group of a free group is not a* CAT(0) *group*, Proc. Amer. Math. Soc. **121** (1994), no. 4, 999–1002.
[41] S. M. Gersten and J. R. Stallings, *Irreducible outer automorphisms of a free group*, Proc. Amer. Math. Soc. **111** (1991), no. 2, 309–314.
[42] Richard Z. Goldstein and Edward C. Turner, *Fixed subgroups of homomorphisms of free groups*, Bull. London Math. Soc. **18** (1986), no. 5, 468–470.
[43] Vincent Guirardel, *Dynamics of* $\mathrm{out}(F_n)$ *on the boundary of Outer space*, Ann. Sci. École Norm. Sup. (4) **33** (2000), no. 4, 433–465.
[44] John L. Harer, *The virtual cohomological dimension of the mapping class group of an orientable surface*, Invent. Math. **84** (1986), no. 1, 157–176.
[45] W. J. Harvey, *Geometric structure of surface mapping class groups*, Homological group theory (Proc. Sympos., Durham, 1977), Cambridge Univ. Press, Cambridge, 1979, pp. 255–269.
[46] Allen Hatcher and Karen Vogtmann, *Isoperimetric inequalities for automorphism groups of free groups*, Pacific J. Math. **173** (1996), no. 2, 425–441.
[47] ———, *Cerf theory for graphs*, J. London Math. Soc. (2) **58** (1998), no. 3, 633–655.
[48] ———, *The complex of free factors of a free group*, Quart. J. Math. Oxford Ser. (2) **49** (1998), no. 196, 459–468.
[49] ———, *Rational homology of* $\mathrm{Aut}(F_n)$, Math. Res. Lett. **5** (1998), no. 6, 759–780.
[50] Gilbert Levitt, *Non-nesting actions on real trees*, Bull. London Math. Soc. **30**

(1998), no. 1, 46–54.

[51] Gilbert Levitt and Martin Lustig, *Irreducible automorphisms of F_n have north-south dynamics on compactified Outer space*, preprint 2002.

[52] ———, *Periodic ends, growth rates, Hölder dynamics for automorphisms of free groups*, Comment. Math. Helv. **75** (2000), no. 3, 415–429.

[53] Martin Lustig, *Structure and conjugacy for automorphisms of free groups I*, Max Planck Institute preprint MPI 2000 - 130.

[54] ———, *Structure and conjugacy for automorphisms of free groups II*, Max Planck Institute preprint MPI 2001 - 4.

[55] Roger C. Lyndon and Paul E. Schupp, *Combinatorial group theory*, Springer-Verlag, Berlin, 1977, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89.

[56] Nataša Macura, *Detour functions and quasi-isometries*, to appear.

[57] ———, *Quadratic isoperimetric inequality for mapping tori of polynomially growing automorphisms of free groups*, Geom. Funct. Anal. **10** (2000), no. 4, 874–901.

[58] Armando Martino and Enric Ventura, *On automorphism-fixed subgroups of a free group*, J. Algebra **230** (2000), no. 2, 596–607.

[59] Zlil Sela, *The Nielsen-Thurston classification and automorphisms of a free group. I*, Duke Math. J. **84** (1996), no. 2, 379–397.

[60] Richard Skora, *Deformations of length functions in groups*, preprint 1989.

[61] John Smillie and Karen Vogtmann, *A generating function for the Euler characteristic of* $\mathrm{Out}(F_n)$, Proceedings of the Northwestern conference on cohomology of groups (Evanston, Ill., 1985), vol. 44, 1987, pp. 329–348.

[62] ———, *Length functions and Outer space*, Michigan Math. J. **39** (1992), no. 3, 485–493.

[63] John R. Stallings, *Topology of finite graphs*, Invent. Math. **71** (1983), no. 3, 551–565.

[64] ———, *Finite graphs and free groups*, Combinatorial methods in topology and algebraic geometry (Rochester, N.Y., 1982), Amer. Math. Soc., Providence, RI, 1985, pp. 79–84.

[65] John R. Stallings and A. Royce Wolf, *The Todd-Coxeter process, using graphs*, Combinatorial group theory and topology (Alta, Utah, 1984), Princeton Univ. Press, Princeton, NJ, 1987, pp. 157–161.

[66] Michael Steiner, *Gluing data and group actions on* $\mathbf{R}$*-trees*, Thesis, Columbia University, 1988.

[67] Edward C. Turner, *Finding indivisible Nielsen paths for a train track map*, Combinatorial and geometric group theory (Edinburgh, 1993), Cambridge Univ. Press, Cambridge, 1995, pp. 300–313.

[68] Enric Ventura, *On fixed subgroups of maximal rank*, Comm. Algebra **25** (1997), no. 10, 3361–3375.

[69] Karen Vogtmann, *Automorphisms of free groups and Outer space*, to appear.

[70] ———, *Local structure of some* $\mathrm{Out}(F_n)$*-complexes*, Proc. Edinburgh Math. Soc. (2) **33** (1990), no. 3, 367–379.

[71] Tad White, *Fixed points of finite groups of free group automorphisms*, Proc. Amer. Math. Soc. **118** (1993), no. 3, 681–688.