THEORY OF MOTIVES, HOMOTOPY THEORY OF VARIETIES, AND DESSINS D'ENFANTS

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1. Introduction

These notes are an expanded version of the transcripts I made during the discussion sessions at the workshop "Theory of motives, homotopy theory of varieties, and dessins d'enfants" held at the American Institute of Mathematics in Palo Alto, California. Thanks go to Gunnar Carlsson and Rick Jardine both for organizing the workshop and for giving me the opportunity to write these notes. Melanie Wood, Layla Pharamond and Bruno Kahn have helped me a lot during the preparation of these notes. I am responsible for all mistakes and errors that remain.

2. Discussion Session One: Open Problems in the Fields

2.1. The Madsen-Tillmann-Weiss equivalence.

Question 2.2 (Gunnar Carlsson). *Is there an algebraic way of analyzing mapping class groups as in the work* [MW02] *of Madsen and Weiss? In other words, is there an algebraic proof of the Mumford conjecture* [Mum83]*?*

The problem is that the process of gluing along the boundaries is a topological, not an algebraic construction. Instead of considering orientable surfaces of genus g with n boundary components, one can consider smooth projective irreducible curves over $\mathbb{C}$ of genus g with n marked points (the collapsed boundaries). Gluing boundaries then corresponds to gluing at marked points – which leads to curves with singularities (nodes). This can be done algebraically, given that one passes from the moduli space $\mathfrak{M}_{g,n}$ of smooth curves to its Deligne-Mumford compactification $\overline{\mathfrak{M}}_{g,n}$, which contains certain singular curves as well. Gluing a disk along a boundary then corresponds to forgetting a marked point, so the result is the Grothendieck's "game of Lego©" on the Teichmüller tower.

Question 2.3 (Marc Levine). *Does compactification lose homotopy information?*

For example, $\mathfrak{M}_{0,4} = \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$ which has nontrivial fundamental group, as opposed to $\overline{\mathfrak{M}}_{0,4} = \mathbb{P}^1$. Here is a program which could supply an answer to 2.2.

(1) Describe gluing on mapping class groups algebraically, to provide an algebraic "game of Lego©".
(2) Use this for an algebraic version of the work of Madsen, Tillmann and Weiss.
(3) Compute the self-maps of this construction and compare it with $G_{\mathbb{Q}}$.

Question 2.4 (Jack Morava)**.** *Does some version of $\widehat{\mathrm{GT}}$ act on the integral cohomology $H^*(B\Gamma_\infty^+; \mathbb{Z})$?*

According to Florian Pop, this is not the case, since $\widehat{\mathrm{GT}}$ is a profinite completion. However, it may act on $H^*(B\Gamma_\infty^+; \mathbb{Z}/n)$. For the latter, one should consider [Gal04].

2.5. **Representations of $\widehat{\mathrm{GT}}$.**

Question 2.6 (Igor Kriz)**.** *Is there a Langlands correspondence between modular functions and representations of $G_{\mathbb{Q}}$?*

The question also makes sense if $G_{\mathbb{Q}}$ is replaced by $\widehat{\mathrm{GT}}$, or rather Λ, the version of $\widehat{\mathrm{GT}}$ incorporating moduli spaces for all genera.

Question 2.7 (Leila Schneps)**.** *Are mixed Tate motives the same as representations of the pro-unipotent completion of $\widehat{\mathrm{GT}}$ (or Λ)? In other words, is the Lie algebra associated to the pro-unipotent completion of $\widehat{\mathrm{GT}}$ the free Lie algebra on odd generators e_{2k+1} of dimension $2k+1 > 1$? These generators correspond to the values $\zeta(2k+1)$ of the Riemann zeta function.*

Remark 2.8. One knows that the Lie algebra associated to $\widehat{\mathrm{GT}}_{\mathrm{pro-unip.}}$ is free [Sch04a]. In this context, one could mention many names, like Deligne, Ihara, Goncharev, Hain, Zagier, ...

2.9. **Cohomology of moduli spaces.**

Question 2.10 (Rick Jardine)**.** *What do we know about the étale or motivic cohomology of the moduli spaces $\mathfrak{M}_g$?*

Faber's conjecture [Fab99] gives an explicit description of the subring of the rational Chow ring $\mathrm{CH}^*(\mathfrak{M}_g) \cong \oplus_{n=o}^{\infty} \mathrm{H}^{n,n}_{\mathrm{mot}}(\mathfrak{M}_g, \mathbb{Q})$ generated by the tautological elements. In fact, one believes that they generate the whole Chow ring. The inclusion

$$\mathfrak{M}_g \hookrightarrow \operatorname*{colim}_{g\to\infty} \mathfrak{M}_g =: \mathfrak{M}_\infty$$

induces a map on singular cohomology

$$H^*(\mathfrak{M}_\infty;\mathbb{Q}) \longrightarrow H^*(\mathfrak{M}_g;\mathbb{Q}). \tag{1}$$

The left hand side of (1) is known to coincide with the rational cohomology of the stable mapping class group, hence

$$H^*(\mathfrak{M}_\infty;\mathbb{Q}) \cong \mathbb{Q}[\kappa_1,\kappa_2,\ldots].$$

The right hand side of (1) is not known explicitly, although certain vanishing results hold at least for the tautological ring [Loo95].

In this respect, it would be interesting to consider the Galois action of $G_\mathbb{Q}$ on the cohomology.

Question 2.11 (Marc Levine). *Does the $G_\mathbb{Q}$-action on $\kappa_i \in H^*(\overline{\mathfrak{M}}_{g,n})$ factor through the cyclotomic character? Does $a \in \hat{\mathbb{Z}}^\times$ act on κ_i by $a^i\kappa_i$? What can be said about the action on the other elements? Is it trivial?*

2.12. **The dreaded tangential basepoint.** Bloch's cycle complex (see [Lev04a]) produces a differential graded algebra whose modules give the category of mixed Tate motives over $\mathbb{Q}$ (in fact, over every number field).

Question 2.13 (Marc Levine). *Is there a homomorphism of differential graded algebras, corresponding to a tangential basepoint in $\mathbb{P}^1 \smallsetminus \{0,1,\infty\}$ at 0, which on cohomology induces the specialization map? In other words, is there a change-of-tangential-basepoint-construction (which is functorial in some derived category) modeling the specialization map?*

According to Bertrand Toen, Markus Spitzweck can do this.

2.14. **The section conjecture.**

Question 2.15 (Leila Schneps). *In how many ways can you find lifts in the diagram*

$$\begin{tikzcd} & \mathrm{Aut}(\pi_1^{\mathrm{alg}}\mathbb{P}^1 \smallsetminus \{0,1,\infty\}) \arrow[d] \\ G_\mathbb{Q} \arrow[r] \arrow[ur, dashed] & \mathrm{Out}(\pi_1^{\mathrm{alg}}\mathbb{P}^1 \smallsetminus \{0,1,\infty\}) \end{tikzcd}$$

where the lower homomorphism is the canonical one (see [Sch04b]*)? More precisely, are all of these lifts given by specifying an element of $\mathbb{P}^1 \smallsetminus \{0,1,\infty\}$ and a tangential basepoint?*

Remark 2.16. One knows that specifying the data mentioned determines such a section (in fact, for any curve). The question is whether all sections arise in this fashion. Among other things, the survey article [NTM01] explains the connection to the Tate conjecture (proven by Faltings in [Fal83]).

Question 2.17 (Marc Levine)**.** *Let $p\colon E \longrightarrow B$ be a fiber bundle and fix a basepoint $e_0 \in E$. Does the homomorphism $\pi_1 E \longrightarrow \pi_1 B$ allow other sections besides those induced by sections $B \longrightarrow E$ of p?*

According to Florian Pop, if the answer is "yes", both base and fiber of the bundle have to be fairly complicated.

2.18. **Maps in étale and motivic cohomology.**

Question 2.19 (Jordan Ellenberg)**.** *Do cohomological constructions built from π_1^{alg} have motivic interpretations? In other words, do maps in étale cohomology come from motivic cohomology"*

According to Marc Levine, this depends on the situation. One problem is that images of motives don't exist in general, since motives don't form an abelian category (see [Lev04b]). For precise maps, see 4.10.

2.20. **p-adic K-theory.**

Question 2.21 (Wieslawa Niziol)**.** *Let p be a prime and let V be a variety over $\hat{\mathbb{Z}}_p$ with smooth generic fiber. How are the K-theories*

$$K(V, \mathbb{Z}/p^n\mathbb{Z}) \quad \text{and} \quad G(V, \mathbb{Z}/p^n\mathbb{Z}) \tag{2}$$

related? If $W \longrightarrow V$ is a blow-up centered in the special fiber, how are the K-theories $K(W)$ and $G(W)$ related?

One knows by [Qui73] that $K(V, \mathbb{Z}/p^n\mathbb{Z}) \cong G(V, \mathbb{Z}/p^n\mathbb{Z})$ if V is regular. A similar statement exists for a blow-up centered in a closed subscheme corresponding to a regular ideal (see [TT90]). From the localization sequence, one can deduce that both the kernel and the cokernel of the map

$$G(V, \mathbb{Z}/p^n\mathbb{Z}) \longrightarrow G(V \otimes_{\hat{\mathbb{Z}}_p} \mathbb{Q}_p, \mathbb{Z}/p^n\mathbb{Z})$$

are annihilated by a single Bott element. So probably K and G differ by a power of p or a Bott element. For the second question, one should mention Haesemeyer's work [Hae03] that homotopy K-theory satisfies cdh-descent for all schemes over a field of characteristic zero.

2.22. **A conjecture from a lecture.** The following conjecture was mentioned in [Sch04b].

Conjecture 2.23 (Parker, 1984)**.** *A Galois dessin D (meaning that the corresponding covering is Galois) is given by a group having two generators a, b. Consider the element*

$$P := \sum\nolimits_{g \in G} (g^{-1}ag, g^{-1}bg)$$

which acts on the rational group ring $\mathbb{Q}[G \times G]$ (considered as a $\mathbb{Q}$-vector space) by right multiplication. Hence there is an associated matrix M_P. The conjecture is that the field of moduli of D (see 4.5) is given by adjoining all the eigenvalues of M_P to $\mathbb{Q}$.

So far, the conjecture has been verified only in abelian cases. Several other conjectures have been mentioned in other lectures, for example the Beilinson-Soulé vanishing conjecture in [Lev04a], Grothendieck's standard conjectures in [Kah04] ....

3. Discussion Session Two: Examples and Speculations

3.1. **Melanie Wood.** An element $\sigma \in G_{\mathbb{Q}}$ has to commute with any map $g\colon X \longrightarrow Y$ of algebraic curves over $\mathbb{Q}$, and hence also with the induced map $\pi_1^{\mathrm{alg}}(g)\colon \pi_1^{\mathrm{alg}}(X) \longrightarrow \pi_1^{\mathrm{alg}}(Y)$. One can write down what it means for σ to commute with $\pi_1^{\mathrm{alg}}(g)$. In particular, if the $G_{\mathbb{Q}}$-action on the algebraic fundamental groups of X and Y is known (for example through the $\widehat{\mathrm{GT}}$-action), one can write down the commutation condition explicitly. This leads to relations that $(\lambda, f) \in G_{\mathbb{Q}}$ must satisfy and defines subgroups between $G_{\mathbb{Q}}$ and $\widehat{\mathrm{GT}}$.

Example 3.2. Let

$$\begin{aligned} g\colon \quad \mathbb{P}^1 &\longrightarrow \mathbb{P}^1 \\ t &\longmapsto -\frac{27}{4}(t^3 - t^2) \end{aligned}$$

be the map corresponding to the cyclic permutation of $\{0, 1, \infty\} \subset \mathbb{P}^1$. One has $g^{-1}(0) = \{0, 1\}$, $g^{-1}(1) = \{-\frac{1}{3}, \frac{2}{3}\}$ and $g^{-1}(\infty) = \{\infty\}$. Ramification occurs only at 0 and 1, hence the induced map

$$\mathbb{P}^1 - g^{-1}(\{0, 1, \infty\}) \longrightarrow \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$$

is unramified. (The associated dessin

is fairly simple.) From the diagram

$$\mathbb{P}^1 \smallsetminus \{0, 1, \infty\} \hookleftarrow \mathbb{P}^1 \smallsetminus g^{-1}(\{0, 1, \infty\}) \xrightarrow{g} \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$$

and the action of (λ, f) on $\pi_1^{\mathrm{alg}}(\mathbb{P}^1 \smallsetminus \{0,1,\infty\})$, one deduces three relations, one of them being the relation

$$f(x^2, y) \equiv y^{-\rho_3} f_\sigma x^{3\rho_3 - 6\rho_2} \bmod y^2, z^3. \tag{3}$$

Here ρ_a is the Kummer cocycle corresponding to a (which depends only on the cyclotomic character λ).

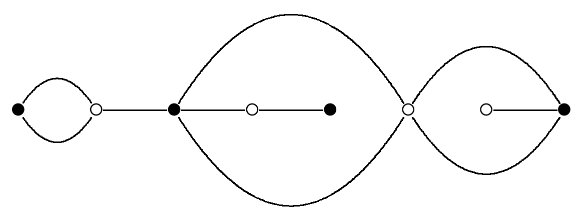
Similarly, we can write down tons more of these relations by considering other functions defined over $\mathbb{Q}$. Many are longer, many follow from others, but plenty don't follow obviously from the relations I, II and III defining $\widehat{\mathrm{GT}}$ that were discussed in [Sch04b]. Since dessins are just conjugacy classes of subgroups of $\pi_1^{\mathrm{alg}}(\mathbb{P}^1 \smallsetminus \{0,1,\infty\})$, relation (3) also gives an invariant on dessins.

Example 3.3. Consider the map

$$\begin{aligned} h\colon \quad \mathbb{P}^1 &\longrightarrow \mathbb{P}^1 \\ t &\longmapsto \frac{27}{4} \frac{t^2(t-1)^2}{(t^2-t+1)^3} \end{aligned}$$

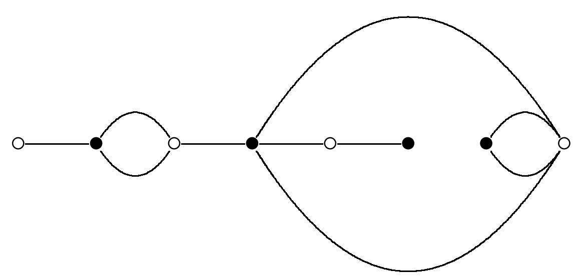
defined over $\mathbb{Q}$. Since σ commutes with h, the dessins

(4)



and

(5)



can be shown to lie in different Galois orbits. The dessins (4) and (5) have the same vertex degree lists, monodromy groups, cartographic groups, automorphism groups, rational Nielsen classes and other known computable invariants. The dessin invariant that separates the Galois orbits is gotten as follows: Take a dessin D (here of genus zero) and the associated Belyi map

$$\beta_D \colon \mathbb{P}^1(\mathbb{C}) \longrightarrow \mathbb{P}^1(\mathbb{C}).$$

Composing with the map h defines a new Belyi map $h(\mathbb{C}) \circ \beta_D$, hence a new dessin $h(D)$. It is obtained from D by copying the dessin of h on every edge of D. Since h is defined over $\mathbb{Q}$, the monodromy group of the new dessin is a $G_{\mathbb{Q}}$-invariant. For the dessin (4), the new dessin has a monodromy group of size $2 \cdot 10^{30}$, while the monodromy group of the dessin derived from dessin (5) has size $2 \cdot 10^{13}$.

Observe that dessin (4) has symmetry with respect to the cyclic permutation $0 \longmapsto 1 \longmapsto \infty \longmapsto 0$. The map g listed in 3.2 is the quotient map having domain $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$ with respect to this symmetry. The map h listed in 3.3 is the quotient map with respect to the full symmetric group Σ_3. The dessins in this example were not constructed systematically, but chosen as the first thing that worked in a search guided by some "symmetry" heuristic.

3.4. **Dan Dugger.** Let k be a field and let Q be either the projective quadric

$$a_1b_1 + a_2b_2 + \ldots + a_mb_m = 0 \quad \text{or} \quad a_1b_1 + a_2b_2 + \ldots + a_mb_m + c^2 = 0$$

in $\mathbb{P}^n$. So either $n = 2m - 1$ or $n = 2m$. Set $DQ_n := \mathbb{P}^n - Q$. The ultimate goal is to compute $H^{*,*}_{\mathrm{mot}}(DQ_n; \mathbb{Z})$. However, very little is known about that. Instead, let's compute $H^{*,*}_{\mathrm{mot}}(DQ_n; \mathbb{Z}/2)$.

Remark 3.5. The computation will imply certain conditions for a sum-of-squares formula in odd characteristic, which was previously known only for characteristic zero. See [DI03] for details.

To get an idea of what the result might be, it is usually helpful to assume $k = \mathbb{C}$ and calculate the singular cohomology of the topological space $DQ_n(\mathbb{C}) = \mathbb{CP}^n \smallsetminus Q(\mathbb{C})$. This space is homotopy equivalent to $\mathbb{RP}^n$. To see this, note that by changing coordinates

$$\begin{aligned} a_j &\longmapsto x_{2j-1} + ix_{2j} \\ b_j &\longmapsto x_{2j-1} - ix_{2j} \\ c &\longmapsto x_{n+2} \qquad \text{(if necessary)} \end{aligned}$$

Q is isomorphic to the projective quadric $x_1 + x_2 + \ldots + x_{n+2} = 0$. In particular, $\mathbb{RP}^n$ embeds in $\mathbb{CP}^n - Q(\mathbb{C})$. The space $\mathbb{CP}^n - Q(\mathbb{C})$ admits a covering of degree two given by the affine quadric $X = a_1b_1 + a_2b_2 + \ldots a_mb_m = 1$. The change of coordinates transforms X into the affine quadric $x_1^2 + \ldots + x_{n+2}^2 = 1$. In particular, S^n embeds in X such that the diagram

$$\begin{tikzcd} S^n \arrow[r, hook] \arrow[d] & X \arrow[d] \\ \mathbb{RP}^n \arrow[r, hook] & \mathbb{CP}^n \smallsetminus Q(\mathbb{C}) \end{tikzcd}$$

commutes, where the vertical maps are the canonical ones. Shrinking the imaginary part of the x_j's then gives a deformation retraction of X to S^n which is compatible with the deck transformation action. In particular, it descends to a deformation retraction of $\mathbb{CP}^n \smallsetminus Q(\mathbb{C})$ to $\mathbb{RP}^n$. Hence we know the singular cohomology of $\mathbb{CP}^n \smallsetminus Q(\mathbb{C})$. However, to lift a computation (there are many) of the singular cohomology of this space to motivic homotopy theory, one should use "the right" computation. Here is one that works. The embedding $Q(\mathbb{C}) \hookrightarrow \mathbb{CP}^n$ induces the following map $H_*(Q(\mathbb{C});\mathbb{Z}) \longrightarrow H_*(\mathbb{CP}^n;\mathbb{Z})$ on singular homology with integer coefficients.

$$\begin{array}{ccccc}
2n & 0 & \xrightarrow{\qquad\qquad\qquad} & \mathbb{Z} \\
2n-1 & \mathbb{Z} & \xrightarrow{\qquad \times 2 \qquad} & \mathbb{Z} \\
\vdots & \vdots & \vdots & \vdots \\
\frac{1}{2}\dim(Q)+1 & \mathbb{Z} & \xrightarrow{\qquad \times 2 \qquad} & \mathbb{Z} \\
\frac{1}{2}\dim(Q) & 0 \text{ or } \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{0 \text{ or fold map}} & \mathbb{Z} \\
\frac{1}{2}\dim(Q)-1 & \mathbb{Z} & \xrightarrow{\qquad \mathrm{id} \qquad} & \mathbb{Z} \\
\vdots & \vdots & \vdots & \vdots \\
1 & \mathbb{Z} & \xrightarrow{\qquad \mathrm{id} \qquad} & \mathbb{Z} \\
0 & \mathbb{Z} & \xrightarrow{\qquad \mathrm{id} \qquad} & \mathbb{Z}
\end{array}$$

Both of the spaces involved are smooth closed manifolds, so by Poincaré duality, one knows the map $H^*(\mathbb{CP}^n;\mathbb{Z}) \longrightarrow H^*(Q(\mathbb{C});\mathbb{Z})$ on singular cohomology. The singular cohomology of $\mathbb{CP}^n - Q(\mathbb{C})$ can then be computed via the Gysin sequence, which ends the discussion of the topological computation.

The integral motivic cohomology of anything is a module over the coefficient ring $H^{*,*}_{\mathrm{mot}}(\mathrm{Spec}(k);\mathbb{Z}) =: \mathbb{M}$, which is big and unknown.

Roughly it looks as follows:

$$
\begin{array}{ccc|cccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & \times & \times & \times & \times & \times & K_3^{\mathrm{Mil}}(k) & 0 & \cdots \\
\cdots & \times & \times \quad q & \times & \times & K_2^{\mathrm{Mil}}(k) & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & k^\times & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & \mathbb{Z} & 0 & 0 & 0 & 0 & \cdots \\
\hline
 & & & & & p \longrightarrow & & & \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{array}
$$

The Beilinson-Soulé vanishing conjecture asserts that the second quadrant is zero. Suppose from now on that $\mathrm{char}(k) \neq 2$ and define $\mathbb{M}_2 := H^{*,*}_{\mathrm{mot}}(\mathrm{Spec}(k); \mathbb{Z}/2)$. There is a short exact sequence

$$
0 \longrightarrow \mathbb{M}_2^{0,1} \longrightarrow k^\times \xrightarrow{\times 2} k^\times \longrightarrow \mathbb{M}^{1,1} \longrightarrow 0 \tag{6}
$$

Hence there is a class $\tau \in \mathbb{M}_2^{0,1} \cong Z/2$ mapping to -1 and a class $\rho \in \mathbb{M}_2^{1,1}$ which is the image of -1. This allows us to state the result of the computation.

Theorem 3.6. *The ring $H^{*,*}_{\mathrm{mot}}(DQ_n; \mathbb{Z}/2)$ is a quotient of the ring $\mathbb{M}_2[a, b]$, where a is a generator of degree $(1, 1)$ and b is a generator of degree $(2, 1)$. If n is even, the relations are $a^2 = \rho a + \tau b$ and $b^m = 0$. If n is odd, there is an additional relation: there is an element $\epsilon \in \mathbb{M}_2^{1,1}$ such that $ab^{m-1} = \epsilon b^{m-1}$*

We will say more about ϵ later.

Remark 3.7. In fact, one can prove that DQ_{2n-1} has the motivic homotopy type of $(\mathbb{A}^n \smallsetminus \{0\})/\{+1, -1\}$. The computation of the motivic cohomology of the latter in the stable case has been used by Voevodsky in the construction of the motivic Steenrod algebra for $p = 2$ (see [Voe03b] and [Voe03a]).

To sketch the proof of 3.6, consider the closed subscheme Z of Q given by $a_1 = 0$. The complement of Z in Q is isomorphic to $\mathbb{A}^{n-1}$. Further, one can show that X is the projective cone on a smaller quadric Q_{-1}. Hence Z has a singular point, the cone point $*$, and the canonical projection $p\colon Z \smallsetminus \{*\} \longrightarrow Q_{-1}$ is a line bundle. In particular, p is a motivic weak equivalence (see 5.2). The closed embedding $i\colon Z \smallsetminus \{*\} \hookrightarrow Q \smallsetminus \{*\}$ of smooth schemes over k leads to a homotopy cofiber sequence

$$\mathbb{A}^{n-1} \cong Q \smallsetminus Z \hookrightarrow Q \smallsetminus \{*\} \longrightarrow \mathrm{Th}(i) \tag{7}$$

by 5.4. Since $\mathbb{A}^{n-1}$ is contractible, the map $Q \smallsetminus \{*\} \longrightarrow \mathrm{Th}(i)$ is a motivic weak equivalence. Thus

$$\begin{aligned} H^{*,*}_{\mathrm{mot}}(Q \smallsetminus \{*\}; Z) &\cong H^{*,*}_{\mathrm{mot}}(\mathrm{Th}(i); \mathbb{Z}) \\ &\cong H^{*-2,*-1}_{\mathrm{mot}}(Z \smallsetminus \{*\}; \mathbb{Z}) \\ &\cong H^{*-2,*-1}_{\mathrm{mot}}(Q_{-1}; \mathbb{Z}) \end{aligned}$$

where the second isomorphism is the Thom isomorphism. Similarly, the closed embedding $\{*\} \hookrightarrow Q$ leads to the homotopy cofiber sequence

$$Q \smallsetminus \{*\} \hookrightarrow Q \longrightarrow \mathbb{A}^{n-1}/\mathbb{A}^{n-1} \smallsetminus \{0\} \tag{8}$$

which induces the Gysin (or localization, or purity) long exact sequence in motivic cohomology. One gets that $H^{*,*}_{\mathrm{mot}}(Q; Z)$ is a free $\mathbb{M}$-module. This corresponds to the decomposition of the motive of Q into Tate motives $\mathbb{Z}(q)$. The last step is the homotopy cofiber sequence

$$\mathbb{P}^n \smallsetminus Q \hookrightarrow \mathbb{P}^n \longrightarrow \mathrm{Th}(c) \tag{9}$$

for the closed embedding $c\colon Q \hookrightarrow \mathbb{P}^n$. The induced map on the motivic cohomology $H^{*,*}_{\mathrm{mot}}(\mathrm{Th}(c); \mathbb{Z}) \longrightarrow H^{*,*}_{\mathrm{mot}}(\mathbb{P}^n; \mathbb{Z})$ is then

$$\begin{array}{ccc} (2n, n) & \mathbb{M} \xrightarrow{\quad\mathrm{id}\quad} \mathbb{M} \\ (2n-2, n-1) & \mathbb{M} \xrightarrow{\quad\mathrm{id}\quad} \mathbb{M} \\ \vdots & \vdots \qquad \vdots \qquad \vdots \\ & \mathbb{M} \xrightarrow{\quad\mathrm{id}\quad} \mathbb{M} \\ (n, \frac{n}{2}) & 0 \text{ or } \mathbb{M} \oplus \mathbb{M} \xrightarrow{0 \text{ or fold map}} \mathbb{M} \\ & \mathbb{M} \xrightarrow{\quad\times 2\quad} \mathbb{M} \\ \vdots & \vdots \qquad \vdots \qquad \vdots \\ (4, 2) & \mathbb{M} \xrightarrow{\quad\times 2\quad} \mathbb{M} \\ (2, 1) & \mathbb{M} \xrightarrow{\quad\times 2\quad} \mathbb{M} \\ (0, 0) & 0 \xrightarrow{\qquad\quad} \mathbb{M}. \end{array}$$

Due to extension problems, we cannot determine the integral motivic cohomology of $\mathbb{P}^n - Q$, since $\mathbb{M}$ might have lots of 2-torsion and 2-cotorsion. In $\mathbb{M}_2$, multiplication with 2 is zero, which immediately gives the additive structure. Extension problems for the multiplicative structure can be resolved using the Bockstein $\beta\colon H^{p,q}_{\mathrm{mot}} \longrightarrow H^{p+1,q}_{\mathrm{mot}}$. Note that $\beta(a) = b$.

Remark 3.8. Concerning the mysterious element $\epsilon \in \mathbb{M}_2^{1,1}$ which appeared in 3.6, note that ab^j is a class of degree $(2j+1, j+1)$. In the topological situation, one can conclude that this class has to vanish if $j = m-1$. This is not possible in the algebraic situation, since the motivic cohomology extends a priori all along the diagonal. If n is even, one can actually prove that ab^j is zero, but if n is odd, one can show only that $ab^{m-1} = \epsilon b^{m-1}$ for some $\epsilon \in \mathbb{M}_2^{1,1}$. Note that if every element of k is a square, then $\mathbb{M}_2^{1,1}$ is zero by (6). A suggestion by Marc Levine is to use étale realizations in order to find out whether ϵ is non-zero in general. Bruno Kahn's suggestion is to use étale descent, since for algebraically closed fields, ϵ is zero.

3.9. **Jack Morava.** The ideas presented here are motivated by Soule's report [Sou99] and the expansion [Sou04]. Certain formulas from representation theory over a finite field $\mathbb{F}_q$ have a well-defined limit when $q \to 1$. For example, the number of $\mathbb{F}_q$-points of a Chevalley group scheme G over $\mathbb{Z}$ converges to the number of points in its Weyl group W. To be more concrete, choosing $G = \mathrm{GL}_\infty$ gives $W = \Sigma_\infty$, the infinite symmetric group. Hence there is a deep connection between the K-theory $K(\mathbb{F}_q) = B\mathrm{GL}_\infty(\mathbb{F}_q)^+$ and stable homotopy $QS^0 = B\Sigma_\infty^+$.

Manin's speculation in this direction (see [Man95]) is that $\mathrm{Spec}(\mathbb{Z})$ should be an affine curve over something absolute, say $\mathrm{Spec}(\mathbb{F}_1)$ (the *field with one element*). This looks like Waldhausen's program concerning the map

$$\mathbb{S} \longrightarrow H\mathbb{Z} \tag{10}$$

of ring spectra. Here $\mathbb{S}$ denotes the sphere spectrum $(S^0, S^1, S^2, \ldots)$ (see [Jar04a]) and $H\mathbb{Z}$ is the Eilenberg-MacLane spectrum of the integers. Waldhausen's goal is to do algebra over $\mathbb{S}$. It seems that the algebraic geometers are trying to invent the sphere spectrum!

In [Sou99], Soulé speculates, motivated by Beilinson's conjecture

$$\mathrm{Ext}_{\mathrm{MTM}_{\mathbb{Z}}}(\mathbb{Z}, \mathbb{Z}(n)) \cong K_{2n-1}(\mathbb{Z}) \tag{11}$$

that there is a category of absolute Tate motives over $\mathbb{F}_1$ such that

$$\mathrm{Ext}_{\mathrm{MTM}_{\mathbb{F}_1}}(\mathbb{Z}, \mathbb{Z}(n)) \cong \pi_{2n-1}(QS^0) = \pi^s_{2n-1}\mathbb{S}. \tag{12}$$

However, the analogy is not that good. There is an algebraic K-theory of ring spectra, defined by Waldhausen [Wal85], which in the special case of the sphere spectrum gives $K(\mathbb{S}) = A(*)$. This K-theory splits as

$$K(\mathbb{S}) \sim QS^0 \times \mathrm{Wh}^{\mathrm{Diff}}(*) \tag{13}$$

with the second factor being the 2-fold delooping of the stable smooth pseudoisotopy space [Wal87]. Another good thing about $K(\mathbb{S})$ is that according to [Wal84] the map (10) induces a rational equivalence on the K-theories, with $K(H\mathbb{Z}) = K(\mathbb{Z})$ (of course).

The correct version of (12) should then be

$$\mathrm{Ext}_{\mathrm{MTM}_{\mathbb{S}}}(\mathbb{Z}, \mathbb{Z}(n)) \cong K_{2n-1}(\mathbb{S})$$

or, more precisely and generally, for any ring spectrum M there should be a spectral sequence

$$\mathrm{Ext}^i_{\mathrm{MTM}_{\mathbb{S}}}(\mathbb{S}, M(q)) \Rightarrow K_{2q-1}(M)$$

of Ext-goups in a category of absolute Tate motives over $\mathbb{S}$.

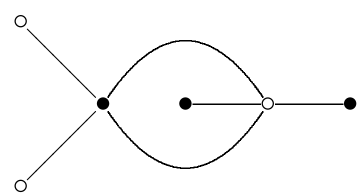
Question 3.10 (Jack Morava). *Is there a category of motives over* $\mathbb{S}$*?*

Another question in this direction is to what extent algebraic geometry can be done over the sphere spectrum. For example, one can say what $\mathbb{P}^n_{\mathbb{S}}$ is and compute its K-theory [Hüt02]. The constructions of that work can be extended to study toric varieties over $\mathbb{S}$. On the other hand, there is the abstract approach [TV03].

4. Discussion Session Three: Examples and Questions

4.1. **Layla Pharamond.** Consider the dessin

(14)



on a surface of genus zero. Since it has six edges, the total ramification degree is six. The list of ramification indices is

0	4,1,1
1	4,1,1
∞	2,4

as one can read off the dessin. From a dessin one can construct a Belyi pair (C, f), where C is a smooth irreducible projective curve over $\overline{Q}$ and $f\colon C(\mathbb{C}) \longrightarrow \mathbb{P}^1(\mathbb{C})$ is a holomorphic function all of whose critical values lie in $\{0, 1, \infty\} \subset \mathbb{P}^1(\mathbb{C})$. The dessin is then given up to isotopy by the preimage under f of the interval

$$0 = \bullet\!\!-\!\!\!-\!\!\circ = 1.$$

The function f is unique up to an automorphism of $\mathbb{P}^1$. In particular, one can choose the function f such that $a \in f^{-1}(a)$ for $a \in \{0, 1, \infty\}$.

Since f is a rational function, the ramification indices imply that

$$f(z) = \lambda \frac{(z-a)^4(z^2+bz+c)}{(z-d)^2(z-e)^4}. \tag{15}$$

Choosing $a = 0$ as the ramification point above 0 of order 4 and ∞ as the ramification point above ∞ of order 4 simplifies (15) to

$$f(z) = \lambda \frac{z^4(z^2+bz+c)}{(z-d)^2}. \tag{16}$$

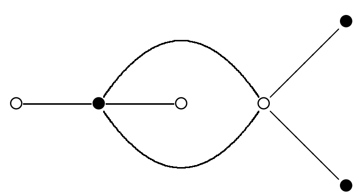
Requiring that 1 is the ramification point above 1 of order 4 implies that $(z-1)^4$ is a factor of $f(z)-1$, which gives a system of four equations with four indeterminates λ, b, c and d. In this example, they can all be expressed in terms of α, where $\alpha^2 = 5$. Choosing the solution $\alpha = \sqrt{5}$ gives the Belyi function

$$f(z) = -\frac{2z^4(2\sqrt{5}z^2 + 2(1-3\sqrt{5})z - 5(1-\sqrt{5}))}{2(\sqrt{5}z - (1+\sqrt{5}))^2} \tag{17}$$

for dessin (14).

Remark 4.2. Choosing the solution $\alpha = -\sqrt{5}$ gives a Belyi function whose dessin is

(18)



The problem of choosing the correct solution for a given dessin is not completely solved.

From the equation (17), one sees immediately that the field of definition (see 4.3) of the function f (and hence of the dessin (14)) is $\mathbb{Q}(\sqrt{5})$. Of course the same is true for (18). Now consider the rational function

$$g(z) = \frac{z^6 - 4z^5 + 20z^3 + 10z^2 + 12z + 2}{25z^2} \tag{19}$$

with field of definition $\mathbb{Q}$. It is ramified over $\{\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \infty\} \in \mathbb{P}^1$, and the preimage $g^{-1}([\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}])$ gives precisely the dessin (14). Hence the field of moduli (see 4.5) of dessin (14) is $\mathbb{Q}$. Perhaps now it is time for precise definitions (see also [CG94]).

Definition 4.3. Suppose a dessin D with Belyi pair (C, f) is given. Let E be a number field and $f_E \colon C_E \longrightarrow B_E$ be a morphism of curves over E. If $f_E \otimes_E \overline{\mathbb{Q}}$ coincides with $f \colon C \longrightarrow \mathbb{P}^1$, then f_E is a *model* of the dessin D, and E is a *field of definition* of D.

Remark 4.4. Although in this case the genus of B_E is zero, it does not have to be $\mathbb{P}^1$. In fact, every conic can appear as the base of a model of a dessin.

If (C, f) is a Belyi pair (hence defined over $\overline{\mathbb{Q}}$) and $\sigma \in G_{\mathbb{Q}}$, base change of f along $\mathrm{Spec}(\sigma) \colon \mathrm{Spec}(\overline{\mathbb{Q}}) \longrightarrow \mathrm{Spec}(\overline{\mathbb{Q}})$ determines a new Belyi pair (C^σ, f^σ). On the level of dessins, write $D \longmapsto D^\sigma$.

Definition 4.5. Let (C, f) be a Belyi pair corresponding to the dessin D. Consider the subgroup M_D of $G_{\mathbb{Q}}$ given by those $\sigma \in G_{\mathbb{Q}}$ for which there exists an isomorphism $u_\sigma \colon C \xrightarrow{\cong} C^\sigma$ and an automorphism $v_\sigma \colon \mathbb{P}^1 \xrightarrow{\cong} \mathbb{P}^1$ such that the diagram

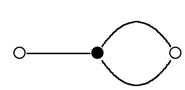
$$\begin{tikzcd} C \arrow[r, "u_\sigma", "\cong"'] \arrow[d, "f"'] & C^\sigma \arrow[d, "f^\sigma"] \\ \mathbb{P}^1 \arrow[r, "v_\sigma", "\cong"'] & \mathbb{P}^1 \end{tikzcd}$$

commutes. That is, f^σ is *weakly isomorphic* to f. The *field of moduli* k_D of D is the fixed field of this subgroup M_D.

If the automorphism v_σ in 4.5 can be chosen as the identity, one says that f^σ is *strongly isomorphic* to f. The fixed field K_D of the subgroup given by those σ is a field of definition. In particular, the field of moduli of a dessin D is contained in the intersection of all fields of definition of D. To prepare a systematic study of the difference between these two fields, let $\Sigma = \Sigma_{0,1,\infty}$ denote the group of permutations of the set $\{0, 1, \infty\}$. Note that every such permutation α gives rise to a unique automorphism $s_\alpha \colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ which is in fact defined over $\mathbb{Q}$. In particular, Σ acts on the set of dessins of genus zero by mapping the Belyi pair (C, f) to the Belyi pair $(C, s_\alpha \circ f)$.

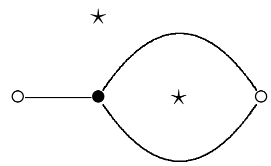
Example 4.6. Consider the permutation $\alpha \in \Sigma$ which fixes 1 and interchanges 0 and ∞, and the dessin D

(20)



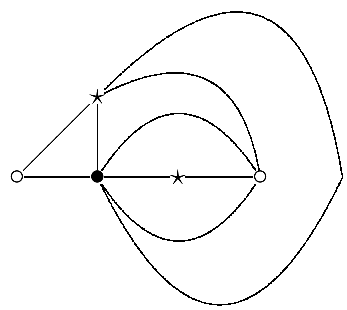
The dessin $\alpha \circ D$ is obtained as follows. First choose a point $\star$ (corresponding to a preimage of ∞) in every open cell of the dessin.

(21)



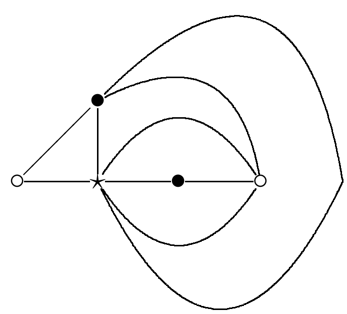
Then connect the $\star$'s via edges with the vertices of the dessin, following the edges of the given dessin in the cyclic fashion determined by the ordering at each vertex.

(22)

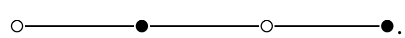


This gives a triangulation of the surface. Now interchange the vertices according to the permutation α

(23)



and forget all $\star$'s, as well as all the edges leading to them. The result is the dessin $\alpha \circ D$



Definition 4.7. Let D be a dessin. Define subgroups

$$\begin{aligned} A_D &:= \{\alpha \in \Sigma | D \text{ is strongly isomorphic to } \alpha \circ D\} \\ B_D &:= \{\alpha \in \Sigma | \exists \sigma \in G_{\mathbb{Q}} \text{ s.t. } D^\sigma \text{ is strongly isomorphic to } \alpha \circ D\} \end{aligned}$$

of the symmetric group. Obviously A_D is a subgroup of B_D, and it is always a normal subgroup.

Proposition 4.8. *The Galois group* $\mathrm{Gal}(K_D/k_D)$ *is canonically isomorphic to the quotient group* B_D/A_D*.*

For a proof of 4.8 consider [Pdd01]. Proposition 4.8 gives a method of deciding whether K_D is different from k_D. For the dessin D in (14), A_D is the trivial subgroup, while B_D consists of the identity permutation and the transposition τ interchanging 0 and 1. Note that $\tau \circ D$ is the dessin (18). Other examples, including a field extension of maximal degree, can be found in [Pdd01].

Remark 4.9. A necessary condition for $K_D \neq k_D$ is that the list of ramification indices agrees for (at least) two points. In case all ramification index lists agree, the Riemann-Hurwitz formula implies that the degree of f is congruent to $2(g-1)$ modulo 3. One can check the possibilities for low degrees; the result is that $K_D \neq k_D$ doesn't happen too often. So far, there is no recipe for producing all examples.

4.10. **Jordan Ellenberg.** Let X be a smooth algebraic curve over a number field k. The curve does not have to be proper. Choosing a base point $x_0 \in X(k)$ gives a short exact sequence

$$\begin{tikzcd} 1 \arrow[r] & \pi_1^{\mathrm{geo}}(X,x_0) \arrow[r] & \pi_1^{\mathrm{alg}}(X,x_0) \arrow[r] & G_k \arrow[r] \arrow[l, bend left, "s_0"] & 1 \end{tikzcd} \tag{24}$$

with splitting induced by the basepoint. For each other $x \in X(k)$, the section $s_x \colon G_k \longrightarrow \pi_1^{\mathrm{alg}}(X,x)$ defines a homomorphism

$$\begin{aligned} X(k) &\longrightarrow H^1(G_k, \pi_1^{\mathrm{geo}}(X,x_0)) \\ x &\longmapsto [s_x s_0^{-1}]. \end{aligned}$$

Remark 4.11. Note that $s_x s_0^{-1}$ is indeed a cocycle. In fact, one should view the homomorphism s_x as having target $\pi_1^{\mathrm{alg}}(X,x_0)$, which is possible up to conjugation induced by changing the basepoint. This conjugation does not affect the cohomology class of $s_x s_0^{-1}$.

The Section Conjecture 2.15 is about the surjectivity of the map

$$X(k) \cup \{\text{tangential basepoints}\} \longrightarrow H^1(G_k, \pi_1^{\mathrm{geo}}(X,x_0)).$$

Later we will see that it should be injective. One problem in the surjectivity is that $\pi_1^{\mathrm{geo}}(X,x_0)$ is too huge. To make it smaller, set

$$\pi' := \pi_1^{\mathrm{geo}}(X,x_0)^{\mathrm{pro}-\ell} \quad \text{and} \quad \pi := \pi' \big/ \big[\pi', [\pi', \pi']\big]. \tag{25}$$

In words, π is the maximal pro-ℓ 2-nilpotent quotient of $\pi_1^{\mathrm{geo}}(X, x_0)$. There is a central extension

$$1 \longrightarrow U_\ell \longrightarrow \pi \longrightarrow \pi^{\mathrm{ab}} = H_1(X, \mathbb{Z}_\ell) \longrightarrow 1 \tag{26}$$

where the ℓ-adic group U_ℓ is a quotient of $\wedge^2\pi^{\mathrm{ab}}$. Using the projection $\pi_1^{\mathrm{geo}}(X, x_0) \longrightarrow \pi$, there results a homomorphism

$$X(k) \longrightarrow H^1(G_k, \pi).$$

If J is the generalized Jacobian of X – so $\pi_1(J) = \pi_1^{\mathrm{ab}}(X)$ – , this homomorphism fits into the commutative diagram

$$\begin{tikzcd}
x \arrow[d, mapsto] & X(k) \arrow[r] \arrow[d] & H^1(G_k, \pi) \arrow[d] \\
{[x - x_0]} & J(k) \arrow[r] \arrow[dr, "\phi"'] & H^1(G_k, \pi^{\mathrm{ab}}) \arrow[d, "\delta"] \\
& & H^2(G_k, U_\ell).
\end{tikzcd}$$

Here the vertical map on the left hand side is the Abel-Jacobi map, whereas the vertical maps on the right hand side come from the central extension (26). To study ϕ and the image of $X(k)$ in $\ker\phi$, observe first that, since U_ℓ is a quotient of $\wedge^2\pi^{\mathrm{ab}}$, the form $\phi(x+y) - \phi(x) - \phi(y) = x \cup y$ is bilinear. Hence if $\ell \neq 2$, ϕ has the form $\phi(x) = \frac{1}{2}(x \cup x) +$ linear term. To what extent is this obstruction useful?

As Marc Levine explained in [Lev04a] and [Lev04b], $H^1(G_k, \pi^{\mathrm{ab}})$ can be thought of as the étale realization of

$$\mathrm{Ext}^1_{\mathrm{mot}}(\mathbb{Z}, H_1(X)) = \mathrm{Hom}_{\mathrm{DM}}(\mathbb{Z}, H_1^{\mathrm{mot}}(X)[1]).$$

Similarly U_ℓ should be the realization of U_{mot}, which in turn is some piece of $H_2^{\mathrm{mot}}(X \times X)$.

Question 4.12 (Jordan Ellenberg)**.** *Is there a motivic map*

$$\mathrm{Ext}^1(\mathbb{Z}, H_1(X)) \xrightarrow{\delta_{\mathrm{mot}}} \mathrm{Ext}^2(\mathbb{Z}, U_{\mathrm{mot}})$$

whose étale realization is δ?

Remark 4.13. According to the Beilinson-Soulé vanishing conjecture, Ext^2 should be zero in the derived category of mixed Tate motives with rational coefficients (see [Lev04b]). However, U_{mot} will in general not be a Tate motive.

Question 4.14. *Could* $\mathrm{Im}\delta_{\mathrm{mot}}$ *(if it exists) or* $\mathrm{Im}\delta$ *be just torsion? Should* $\mathrm{Ext}^2(\mathbb{Z}, M)$ *be zero also if M is not a Tate motive?*

Example 4.15. As an example, consider $X = \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$ with a tangential basepoint. The Jacobian is $J = \mathbb{G}_m^2$, $\pi^{\mathrm{ab}} = (\mathbb{Z}_\ell(1))^2$, π is a Heisenberg group and $U_\ell = \mathbb{Z}_\ell(2)$. Since Ext^2 here is the torsion of K_2 of a number ring, it vanishes rationally. The Abel-Jacobi map has the form

$$\begin{array}{ccc} X(k) = k \smallsetminus \{0,1\} & \longrightarrow & k^\times \times k^\times = J(k) \\ x & \longmapsto & (x, 1-x) \end{array}$$

and ϕ maps (a, b) to the symbol $\{a, b\} \in K_2^{\mathrm{Mil}}(k)/\ell = H^2(G_k, \mathbb{Z}/\ell(2))$. Note that here we switched from $\mathbb{Z}_\ell$-coefficients to $\mathbb{Z}/\ell$-coefficients, since Florian Pop pointed out that $H^2(G_k, \mathbb{Z}_\ell(2)) = 0$.

There are cases where one can show something about curves over local fields. In particular, the obstruction given by ϕ can be non-zero. The relation between δ and the Neron-Tate map for the local part deserves to be worked out. Going more steps down the lower central series gives further obstructions which are related to Massey products.

Question 4.16 (Marc Levine)**.** *How can the central extension (26) be done motivically?*

4.17. **Bruno Kahn.** Let X be a smooth affine curve over $\mathbb{C}$. Then the topological space $X(\mathbb{C})$ is an Eilenberg-MacLane space $K(\pi, 1)$ for some group π.

Question 4.18 (Bruno Kahn)**.** *Suppose X is a smooth affine curve over a subfield $k \hookrightarrow \mathbb{C}$ of the complex numbers. What is the relation between X regarded as an object in the motivic stable homotopy category* $\mathrm{SH}(k)$ *and the classifying space $B\pi_1(X(\mathbb{C}))$?*

The origin of this question is part 3 of [Voe00, 3.4.2]. Note that the étale realization of such an X is the classifying space for $\pi_1^{\mathrm{alg}}(X)$. To understand what might happen in $\mathrm{SH}(k)$, consider the motive $M(X) \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$ first. Suppose (for simplicity, it works also in general) that $x \in X(k)$ is a rational point. Then there exists a morphism $X \longrightarrow J$, where J is the generalized Jacobian of X. One can show that J is homotopy-invariant and has transfers. Hence there exists a lift in the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & J \\ L(X) \arrow[ur, dashed] & \end{tikzcd}$$

where $L(X)$ is the representable sheaf with transfers generated by X (see for example [Voe98] or [Lev04a]). By definition, $M(X) = C_*L(X)$

is the Suslin complex of $L(X)$ using the cosimplicial scheme considered for example in [Lev04a]. Since J is homotopy-invariant, one has $C_*(J) \cong J$. The result is a map $M(X) \longrightarrow J$ which is basically an isomorphism, since X is an affine curve. Details can be found in [SS03].

On topological fundamental groups, the map $X \longrightarrow J$ induces the map $\pi_1 X(\mathbb{C}) \longrightarrow \pi_1 J(\mathbb{C}) = \pi_1^{\mathrm{ab}} X(\mathbb{C})$. That is, we can interpret $\pi_1^{\mathrm{ab}}(X)$ in terms of motives. What about the whole π_1?

Lemma 4.19. *Let G be a free group, set $G_0 := G$ and let $G_{n+1} := [G_n, G_n]$ be the terms in the derived series. The canonical map of spectra*

$$\Sigma^\infty BG \longrightarrow \operatorname*{holim}_n \Sigma^\infty B(G/G_n)$$

is a weak equivalence.

Proof. To prove this, we will calculate the map on the level of stable homotopy groups. Using the Atiyah-Hirzebruch spectral sequence, it suffices to prove that for any abelian group A (that one should think of as $\pi_i^s(S^0)$)

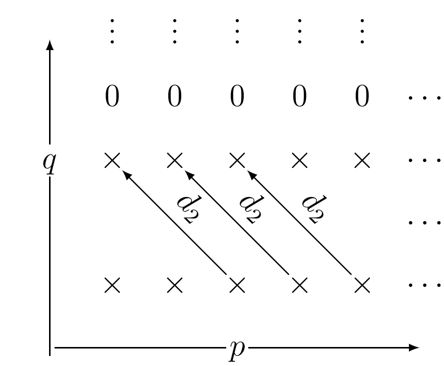
$$H_i(G, A) \cong \lim_n H_i(G/G_n, A) \tag{27}$$

$$\lim\nolimits^1 H_i(G/G_n, A) \cong 0 \tag{28}$$

for all i. In fact, it suffices to prove this for $A = \mathbb{Z}$. The Hochschild-Serre spectral sequence

$$H_p(G/G_n, H_q(G_n, \mathbb{Z})) \Rightarrow H_{p+q}(G, \mathbb{Z}) \tag{29}$$

is concentrated in the first quadrant and has the form



since G is a free group. Most of the d_2 differentials are isomorphisms. To be more precise, the differential

$$H_{p+2}(G/G_n, \mathbb{Z}) \xrightarrow{d_2} H_p(G/G_n, G_n^{\mathrm{ab}})$$

is an isomorphism for $p \geq 1$. For $p = 0$ the sequence

$$0 \longrightarrow H_2(G/G_n, \mathbb{Z}) \longrightarrow H_0(G/G_n, G_n^{\mathrm{ab}}) \longrightarrow G^{\mathrm{ab}} \longrightarrow (G/G_n)^{\mathrm{ab}} \longrightarrow 0$$

is exact and the diagram

$$\begin{array}{ccc} H_{p+2}(G/G_{n+1}, \mathbb{Z}) & \xrightarrow{d_2} & H_p(G/G_{n+1}, G_{n+1}^{\mathrm{ab}}) \\ \downarrow & & \downarrow \\ H_{p+2}(G/G_n, \mathbb{Z}) & \xrightarrow{d_2} & H_p(G/G_n, G_n^{\mathrm{ab}}) \end{array}$$

commutes. Now the homomorphism $G_{n+1}^{\mathrm{ab}} \longrightarrow G_n^{\mathrm{ab}}$ is trivial, which implies that

$$H_i(G/G_{n+1}, \mathbb{Z}) \longrightarrow H_i(G/G_n, \mathbb{Z})$$

is trivial for $i \geq 2$. It follows that the isomorphisms (27) and (28) hold for $i \geq 2$. If $i = 1$, the homomorphism $G^{\mathrm{ab}} \longrightarrow (G/G_n)^{\mathrm{ab}}$ is an isomorphism for $n \geq 1$, which concludes the proof. $\square$

Suppose now that G is finitely generated and free, so $G = F_n$ with generators $\{e_1, \ldots, e_n\}$. Then G_1/G_2 is generated by the commutators $[e_i, e_j]$ over the group algebra G/G_1. Unfortunately, lemma 4.19 does not hold for the lower central series, because it doesn't go down as fast as the derived series. As Florian Pop pointed out, there is no term in the lower central series which is contained in G_2, since it is much too small.

Question 4.20 (Bruno Kahn)**.** *Can you find a motivic structure on G_1/G_2 and on the next quotients?*

Remark 4.21. According to a conjecture of Ihara, G_1/G_2 is free on the commutators $[e_1, e_j]$ over the group algebra $\mathbb{Z}[G/G_1]$. Hence the motivic structure – if it exists – has weights in every dimension.

Suppose that $X \longrightarrow \mathrm{Spec}(\mathbb{C})$ is a *good neighborhood* in the sense of Artin (see [SGA73, Exp. XI]). Then $X(\mathbb{C})$ is a $K(\pi, 1)$ such that π is a successive extension of free groups. In particular, π is a *good group* in the sense of Serre. Lemma 4.19 is motivated by Serre's proof of Artin's comparison theorem (see [SGA73, Exp. XI, Variante 4.6]). However, it is not clear whether lemma 4.19 will be true for good groups with the derived series. Maybe one has to use another filtration.

Question 4.22 (Bertrand Toen)**.** *As Bruno Kahn explained, the abelian part of the motivic fundamental group is the Jacobian. Is there a way to realize the full motivic fundamental group without using étale realization?*

5. DISCUSSION SESSION FOUR: EXAMPLES

5.1. **Dan Isaksen.** Here are a few concrete and standard examples of motivic homotopy calculations over a field k. To understand these, the precise meaning of *motivic space over* k (a.k.a. "simplicial presheaf on Sm/k") which was discussed in [Jar04a], is not so important. Instead, one can mostly consider only objects in Sm/k (that is, smooth schemes of finite type over k) and formal quotients (or colimits) of these. One example is $\mathbb{A}^1/\mathbb{A}^1 \smallsetminus \{0\}$, the Thom space (A.5) of the trivial line bundle over k. However, by taking formal colimits of smooth schemes over k, one forgets the (few) colimits that already exist in Sm/k. For example, if $X \in \mathrm{Sm}/k$ is covered by two open subschemes U and V, the map f appearing in the diagram

$$\begin{tikzcd}
U \cap V \arrow[r, hook] \arrow[d, hook] & V \arrow[d] \arrow[ddr, hook] & \\
U \arrow[r, hook] \arrow[drr, hook] & U \coprod_{U\cap V} V \arrow[dr, "f"] & \\
& & X
\end{tikzcd}$$

is in general not an isomorphism. This is what the *local weak equivalences* are for: the map f is a local weak equivalence for the Zariski (and hence the Nisnevich) topology. In other words, although the square

$$\begin{tikzcd}
U \cap V \arrow[r, hook] \arrow[d, hook] & V \arrow[d, hook] \\
U \arrow[r, hook] & X
\end{tikzcd}$$

is not a pushout square of motivic spaces in general, it is a homotopy pushout square (A.7). An example of a Nisnevich local weak equivalence which is not a Zariski local weak equivalence is the map g in the diagram

$$\begin{tikzcd}
\mathbb{A}^1_{\mathbb{C}} \smallsetminus \{(X-i),(X+i)\} \arrow[r, hook, "\circ"] \arrow[d] & \mathbb{A}^1_{\mathbb{C}} \smallsetminus \{(X-i)\} \arrow[d] \arrow[ddr] & \\
\mathbb{A}^1_{\mathbb{R}} \smallsetminus \{(X^2+1)\} \arrow[r] \arrow[drr, hook, "\circ"] & \text{pushout} \arrow[dr, "g"] & \\
& & \mathbb{A}^1_{\mathbb{R}}.
\end{tikzcd}$$

Here an irreducible polynomial is identified with the closed point it determines in $\mathbb{A}^1_k$. The *motivic weak equivalences* are obtained from the Nisnevich local weak equivalences by requiring that $\mathbb{A}^1$ is contractible. Vector bundles and, more generally, $\mathbb{A}^1$-homotopy equivalences are motivic weak equivalences.

Example 5.2. Let $p\colon E \longrightarrow B$ be a vector bundle in Sm/k. Then p is an $\mathbb{A}^1$-homotopy equivalence, with homotopy inverse given by the zero section $s\colon B \hookrightarrow E$. The composition $p \circ s$ is the identity, and the map $H\colon E \times \mathbb{A}^1 \longrightarrow E$ which is locally defined by $(x,t) \longmapsto tx$ is an $\mathbb{A}^1$-homotopy from $s \circ p$ to id_E. As a consequence, p is a motivic weak equivalence. Here, B is even a strong $\mathbb{A}^1$-deformation retract of E.

Another important example is the $\mathbb{A}^1$-homotopy between the cyclic permutation and the identity map on $\mathbb{A}^3/\mathbb{A}^3 \smallsetminus \{0\}$ which appeared in Jardine's lecture [Jar04a].

Example 5.3. Set $S^{1,0} := S^1$ and $S^{1,1} := \mathbb{G}_m$. In general, define $S^{p,q} := S^{p-q} \wedge \mathbb{G}_m^{\wedge q}$ where $p \geq q$. Since $\mathbb{A}^1$ is contractible, there are weak equivalences

$$\mathbb{A}^1/\mathbb{A}^1 \smallsetminus \{0\} \sim S^1 \wedge \mathbb{G}_m = S^{2,1} \sim \mathbb{P}^1$$

where $\mathbb{P}^1$ is pointed by 1. For the latter weak equivalence, one uses the canonical covering

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r, hook, "\circ"] \arrow[d, hook, "\circ"] & \mathbb{A}^1 \arrow[d, hook, "\circ"] \\
\mathbb{A}^1 \arrow[r, hook, "\circ"] & \mathbb{P}^1.
\end{tikzcd}$$

Similarly, one can use the covering

$$\begin{tikzcd} \mathbb{G}_m \times \mathbb{G}_m \arrow[r, hook] \arrow[d, hook] & \mathbb{A}^1 \times \mathbb{G}_m \arrow[d, hook] \\ \mathbb{G}_m \times \mathbb{A}^1 \arrow[r, hook] & \mathbb{A}^2 \smallsetminus \{0\} \end{tikzcd}$$

to conclude that $\mathbb{A}^2 \smallsetminus \{0\} \sim S^{3,2}$. By induction, one gets the Thom spaces of the trivial bundles over a point

$$\mathbb{A}^n/\mathbb{A}^n \smallsetminus \{0\} \sim S^{2n,n}. \tag{30}$$

Concerning the indexing of the sphere $S^{p,q}$, p is the dimension and q is the Tate twist. One can read off these numbers if $k = \mathbb{R}$ by taking complex points, together with the $\mathbb{Z}/2$-action of complex conjugation. Then p is the dimension of the sphere and q is the number of coordinate axes where the $\mathbb{Z}/2$-action is nontrivial.

Another important ingredient for calculations in motivic homotopy theory is the so-called "Homotopy Purity Theorem" of Morel and Voevodsky [MV99, 2.23].

Theorem 5.4. *Let $i\colon Z \hookrightarrow X$ be a closed embedding in* Sm/k, *and let $p\colon N \longrightarrow Z$ be the normal bundle of i. Then there are motivic weak equivalences connecting $X/X \smallsetminus i(Z)$ and the Thom space* $\mathrm{Th}(p)$.

Example 5.5. Consider $\mathbb{P}^1 \smallsetminus \{0,1,\infty\} = \mathbb{A}^1 \smallsetminus \{0,1\}$. It is an open subscheme of $\mathbb{A}^1$, and by 5.4, the quotient $\mathbb{A}^1/\mathbb{A}^1 \smallsetminus \{0,1\}$ is weakly equivalent to the Thom space of the normal bundle associated to the closed embedding

$$i\colon \mathrm{Spec}(k) \coprod \mathrm{Spec}(k) = \{0,1\} \hookrightarrow \mathbb{A}^1.$$

The normal bundle is trivial in this case, hence the Thom space is

$$\{0,1\} \times \mathbb{A}^1/\{0,1\} \times (\mathbb{A}^1 \smallsetminus \{0\}) \cong \left(\mathbb{A}^1/\mathbb{A}^1 \smallsetminus \{0\}\right) \vee \left(\mathbb{A}^1/\mathbb{A}^1 \smallsetminus \{0\}\right).$$

On the other hand, since $\mathbb{A}^1$ is contractible, the quotient $\mathbb{A}^1/\mathbb{A}^1 \smallsetminus \{0,1\}$ is weakly equivalent to the unreduced suspension of $\mathbb{A}^1 \smallsetminus \{0,1\}$. If $k \neq \mathbb{F}_2$, one can choose a basepoint in $\mathbb{A}^1 \smallsetminus \{0,1\}$, which gives weak equivalences

$$S^1 \wedge \mathbb{P}^1 \smallsetminus \{0,1,\infty\} \sim (\mathbb{A}^1/\mathbb{A}^1 \smallsetminus \{0\}) \vee (\mathbb{A}^1/\mathbb{A}^1 \smallsetminus \{0\}) \sim S^1 \wedge (\mathbb{G}_m \vee \mathbb{G}_m).$$

Hence if we consider motivic spectra instead of motivic spaces, where smashing with S^1 is invertible (up to weak equivalence), there results a stable equivalence

$$\mathbb{P}^1 \smallsetminus \{0,1,\infty\} \sim \mathbb{G}_m \vee \mathbb{G}_m. \tag{31}$$

This implies that the motivic cohomology of $\mathbb{P}^1 \smallsetminus \{0,1,\infty\}$ has the form

$$H^{*,*}_{\mathrm{mot}}(\mathbb{P}^1 \smallsetminus \{0,1,\infty\}) \cong H^*_{\mathrm{sing}}(S^1 \vee S^1) \otimes H^{*,*}_{\mathrm{mot}}(\mathrm{Spec}(k)).$$

Since the property of having a k-point is invariant under motivic weak equivalence [MV99, 2.5], relation (31) can not hold unstably if $k = \mathbb{F}_2$.

Appendix A. Definitions and Notation

Notation A.1. If $E \hookrightarrow F$ is a field extension, $\mathrm{Gal}(F/E)$ denotes its Galois group. The absolute Galois group of a field k is denoted G_k. We are particularly interested in $G_{\mathbb{Q}}$.

Definition A.2. A *Belyi pair* (X, f) consists of a smooth irreducible projective complex curve X, together with a holomorphic function $f\colon X(\mathbb{C}) \longrightarrow \mathbb{P}^1(\mathbb{C})$ which is unramified outside of $\{0,1,\infty\}$. By Belyi's theorem, X is then actually defined over $\overline{\mathbb{Q}}$ (see [Sch94]).

Notation A.3. Given a Belyi pair (X, f), the preimage $f^{-1}([0,1]) \subset X(\mathbb{C})$ determines a dessin (see [Sch04b]). We use the convention that a preimage of zero is denoted $\bullet$, a preimage of 1 is denoted $\circ$, and a preimage of ∞ is denoted $\star$.

Notation A.4. Usually an open embedding of schemes is denoted $X \hookrightarrow\!\!\circ\!\!\rightarrow Y$, and a closed embedding is denoted $X \hookrightarrow\!\!\!+\!\!\!\rightarrow Y$.

Definition A.5. Let $p\colon E \longrightarrow B$ be a (topological or algebraic) vector bundle, with zero section s. If p is algebraic, the *Thom space* of p is the quotient $E/E - s(B)$ (viewed as a pointed presheaf of some category of schemes). If p is topological, the *Thom space* of p is the quotient $D(p)/S(p)$, where $D(p)$ (resp. $S(p)$) denotes the associated disk (resp. sphere) bundle of p. In both cases, the Thom space is the homotopy-theoretical meaningful collapse of the complement of the zero section.

Definition A.6. Let S^1 denote the simplicial circle $\Delta^1/\partial\Delta^1$, as well as its geometric realization (which is homeomorphic to the space of complex numbers having norm one). If X is a space with basepoint x_0, the space of loops in X based at x_0 is denoted $\Omega(X, x_0)$. In order to make homotopical sense out of this in case X is a simplicial set, embed X in a weakly equivalent Kan simplicial set X' and set $\Omega(X, x_0) := \mathbf{sSet}_*(S^1, (X', x_0))$. The right hand side denotes the pointed simplicial set having as its n-simplices the pointed maps from $S^1 \wedge \Delta^n_+$ to X'.

If $E = (E_0, E_1, \ldots)$ is a spectrum [Jar04a], denote by $\Omega^\infty E$ the colimit of the sequence

$$E_0 \longrightarrow \Omega E_1 \longrightarrow \Omega^2 E_2 \longrightarrow \cdots.$$

A space is an *infinite loop space* if it is homotopy equivalent to $\Omega^\infty E$ for some spectrum E.

Definition A.7. A commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$

of simplicial sets or topological spaces or motivic spaces is a *homotopy pushout square* if the following holds. Factor f as a decent inclusion (a *cofibration*) $i\colon X \longrightarrow T$ followed by a weak equivalence $T \xrightarrow{\sim} Y$. Then the induced map $Z \coprod_X T \longrightarrow W$ is also a weak equivalence. In the case of simplicial sets or motivic spaces, any inclusion is decent. For example, if X is pointed, the square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & S^1 \wedge X \end{tikzcd}$$

is a homotopy pushout square. A *homotopy cofiber sequence* is a homotopy pushout square as above in which $Z = *$.

Notation A.8. Let k be a field. The derived category of effective geometrical motives over k, as described in [Lev04a], is denoted $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$. If one inverts tensoring with the Tate motive $\mathbb{Z}(1)$ in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$, one obtains the derived category of geometrical motives $\mathrm{DM}(k)$.

A.9. **Background on the Mumford conjecture.** Let $\Gamma_{g,b}$ denote the *mapping class group* of an oriented surface $F_{g,b}$ of genus g with b boundary components. That is, $\Gamma_{g,b} := \pi_0\mathrm{Diff}(F_{g,b}, \partial)$. In [Mum83], Mumford conjectured that the rational group cohomology of $\Gamma_{g,b}$ coincides with the polynomial algebra $\mathbb{Q}[\kappa_1, \kappa_2, \ldots]$ in degrees less than $\frac{g-1}{2}$. Here κ_i is the i-th Miller-Morita-Mumford class having degree $2i$.

There is a canonical homomorphism $\Gamma_{g,b} \longrightarrow \Gamma_{g+1,b}$ induced by glueing on a torus with two boundary components, as well as a homomorphism $\Gamma_{g,b} \longrightarrow \Gamma_{g,b-1}$ (for b positive) induced by glueing on a disk. These homomorphisms give maps of classifying spaces [Jar04b] $B\Gamma_{g,b} \longrightarrow B\Gamma_{g+1,b}$ resp. $B\Gamma_{g,b} \longrightarrow B\Gamma_{g,b-1}$, which induce isomorphisms in integral cohomology in degrees less than $\frac{g-1}{2}$ by the stability theorems of Harer [Har85] and Ivanov [Iva89]. Hence one can compute the integral group cohomology of $\Gamma_{g,b}$ in degrees less than $\frac{g-1}{2}$ via the integral cohomology of the space $B\Gamma_\infty$ appearing as the colimit of the sequence

$$B\Gamma_{g,1} \longrightarrow B\Gamma_{g+1,1} \longrightarrow B\Gamma_{g+2,1} \longrightarrow \cdots .$$

Since Quillen's plus-construction does not change the cohomology, one can equally use the space $B\Gamma_\infty^+$, which (as well as $\mathbb{Z} \times B\Gamma_\infty^+$) Ulrike Tillmann proved to be an infinite loop space (A.6) in [Til97]. Since the infinite loop space structure is somewhat surprising, Tillman and Madsen conjectured that it comes from a different infinite loop space, which can be described as follows.

Let $\mathrm{Gr}(2,n)$ be the Grassmann manifold of oriented 2-planes in $\mathbb{R}^{n+2}$. It is the base for two canonical bundles, the tautological 2-plane bundle and the orthogonal complement n-bundle $L_n \longrightarrow \mathrm{Gr}(2,n)$. The Thom spaces (A.5) of the bundles $L_0, L_1, L_2, \ldots$ form a spectrum $\mathbb{CP}^\infty_{-1} := (\mathrm{Th}(L_0), \mathrm{Th}(L_1), \ldots, \mathrm{Th}(L_n), \ldots)$. Hence there is an infinite loop space $\Omega^\infty\mathbb{CP}^\infty_{-1}$. Relying on work of Madsen and Tillman [MT01], Madsen and Weiss proved the following in [MW02].

Theorem A.10. *There is a homotopy equivalence*

$$\mathbb{Z} \times B\Gamma_\infty^+ \longrightarrow \Omega^\infty\mathbb{CP}^\infty_{-1}.$$

Since the rational cohomology of any path component of the target is equal to the rational cohomology of BU, where U is the infinite unitary group, Mumford's conjecture follows.

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