Notes on Creation Operators

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August 4, 2005

This set of notes will cover sort of a 'how to' about creation operators. Because there were several references in later talks, the appendix will cover creation operators for 'generalized Kostka polynomials' including programs in Maple (using John Stembridge's `SF` package) and MuPAD (using the `combinat` library).

A set of creation operators for a basis $\{b_\lambda\}$ of the space $\Lambda = \mathbb{Q}[p_1, p_2, p_3, \ldots]$ which is indexed by the set of partitions is a family of operators $\{B_m\}_{m\geq 0}$ where $B_m \in End(\Lambda)$ such that $B_{\lambda_1} B_{\lambda_2} \cdots B_{\lambda_\ell} 1 = b_\lambda$.

Notice that this condition is not sufficient to completely determine a unique family B_m because it does not determine the action of B_m on a basis. We have

$$B_m(b_\lambda) = b_{(m,\lambda_1,\ldots,\lambda_\ell)} \text{ if } m \geq \lambda_1$$

but we have not said what the action of B_m is on b_λ for $m < \lambda_1$. For this reason we could have many families of creation operators for the basis b_λ and only some of them are 'nice' (i.e. have some sort of natural expression or have relatively simple combinatorial action on another basis).

Why should we look at creation operators?

- Understanding B_m tells us how to build larger basis elements from smaller ones. If we know the action of B_m on another basis we get a recursive method for expanding b_λ in terms of that basis (ideally giving a way of obtaining a combinatorial interpretation for the coefficients).

  Example 1 *Let B_m = multiplication by the Schur function s_m. The family $\{B_m\}$ creates the homogeneous basis. $B_m(s_\lambda)$ is given by the Pieri rule. This implies that the homogeneous basis when expanded in terms of the Schur functions is given by $b_\lambda = \sum_\mu K_{\mu\lambda} s_\mu$ where the coefficients $K_{\mu\lambda}$ are the column strict tableaux of shape μ and content λ. We could hope that a similar attack might work on more complicated bases (e.g. the Macdonald basis).*

- Given a formula for B_m we can use it as a definition of the basis b_λ and derive properties about it (e.g. triangularity relations, duality, positivity or polynomiality of change of basis coefficients, etc.).

Example 2 *A Macdonald creation operator. In 1995, there were several proofs that came out roughly at the same time of the polynomiality of the Macdonald q,t-Kostka polynomials and at least two of these used the approach of creation operators. The formulas for creation operators for the Macdonald polynomials followed naturally from a similar idea of Lapointe for Jack polynomials. (Kirillov-Noumi [KN1] [KN2] Lapointe-Vinet [LV1] [LV2]) Set $T_{q,x_i}P[X_n] = P[X_n - (1-q)x_i]$, and for a subset $J \subseteq \{1,2,\ldots,n\}$ define*

$$D_J^r(q,t) = \sum_{\substack{I \subseteq J \\ |I|=r}} A_I(X_n;t) \prod_{i \in I} T_{q,x_i}$$

where $A_I(X_n;t) = t^{\binom{r}{2}} \prod_{i \in I, j \notin I} \frac{tx_i - x_j}{x_i - x_j}$. Now set, $D_I(u;q,t) = \sum_{r=0}^n u^r D_I^r$.

$$B_m = \sum_{|I|=m} \sum_{r=0}^{m} t^{-r} x_I \prod_{\substack{i \in I \\ j \notin I}} \frac{x_i - x_j/t}{x_i - x_j} D_I^r(-t;q,t)$$

They proved that B_m adds a column to the basis $J_\lambda[X_n;q,t]$ and concluded that since the operator B_m didn't introduce denominators of the form $1/t$ or $1/q$ that the coefficients $K_{\mu\lambda}(q,t)$ must be polynomials in q and t.

There are several methods for getting a handle on creation operators.

- Find the action on a basis $\{d_\lambda\}$ by expressing d_μ in the basis $\{b_\lambda\}$. Since $B_m(b_\lambda) = b_{(m,\lambda_1,\ldots,\lambda_\ell)}$, then

$$d_\mu = \sum_\lambda c_{\mu\lambda} b_\lambda$$

so

$$B_m(d_\mu) = \sum_\lambda c_{\mu\lambda} B_m(b_\lambda).$$

If $c_{\mu\lambda} = 0$ for $\lambda_1 \geq \mu_1$, then the action of B_m on d_μ for $\mu_1 \leq m$ is uniquely determined.

Example 3 *Let B_m create the Schur basis. The action of B_m on the $\{m_\lambda\}$-basis can be computed explicitly if $m < \lambda_1$. For example we can calculate B_3 on m_λ for $|\lambda| = 3$ as*

$$B_3(m_{(3)}) = s_{(3,3)} - s_{(3,2,1)} + s_{(3,1,1,1)} = m_{3,3} - m_{2,2,2} - m_{2,2,1,1} - m_{2,1,1,1,1} - m_{1,1,1,1,1,1}$$

$$B_3(m_{(2,1)}) = s_{(3,2,1)} - 2s_{(3,1,1,1)} = m_{(3,2,1)} + 2m_{(2,2,2)} + 2m_{(2,2,1,1)} - 4m_{(1,1,1,1,1,1)}.$$

$$B_3(m_{(1,1,1)}) = s_{(3,1,1,1)} = m_{(3,1,1,1)} + m_{(2,2,1,1)} + 4m_{(2,1,1,1,1)} + 10m_{(1,1,1,1,1,1)}$$

If this recurrence is 'nice' for this basis then we can hope to recognize the action. Understanding this particular recurrence completely could lead to another combinatorial interpretation for $K_{\lambda\mu}$ since it is the coefficient of m_λ in s_μ.

- The operators $\{s_\lambda \circ s_\mu^\perp\}$ form a basis for $End(\Lambda)$ where s_λ represents multiplication by the Schur function s_λ, $s_\lambda(s_\nu) = \sum_\gamma c^\gamma_{\lambda\nu} s_\gamma$ and $s_\mu^\perp$ is the operation which is dual to multiplication by s_μ, that is, $s_\mu^\perp(s_\gamma) = \sum_\nu c^\gamma_{\nu\mu} s_\lambda$. One way of finding an expression for B_m in terms of this basis is to determine what it must be up to a certain degree by acting on a basis of Λ where the action is known and adding or subtracting terms and then guessing at the formula.

  Example 4 *Let B_m again create the Schur basis. To find a formula for B_3 we know that $B_3(1) = s_{(3)}$ so $B_3 = s_{(3)} + X$ where X are other terms which involve $s_\lambda \circ s_\mu^\perp$ with $|\mu| > 0$. Next try $B_3(s_{(1)}) = s_{(3,1)}$, and $(s_{(3)} + X)(s_{(1)}) = s_{(3,1)} + s_{(4)} + X(s_{(1)})$ so $X(s_{(1)}) = -s_{(4)}$ and it must be that $X = -s_{(4)} s_{(1)}^\perp + X'$ where X' involves terms $s_\lambda \circ s_\mu^\perp$ with $|\mu| > 1$ so $B_3 = s_{(3)} - s_{(4)} s_{(1)}^\perp + X'$. Now act by this on $s_{(2)}$ and $s_{(1,1)}$ and determine that $X'(s_{(2)}) = 0$ and $X'(s_{(1,1)}) = -s_{(5)}$, so we know that $B_3 = s_{(3)} - s_{(4)} s_{(1)}^\perp + s_{(5)} s_{(1,1)}^\perp + X''$ where X'' involves terms $s_\lambda \circ s_\mu^\perp$ with $|\mu| > 2$. There are several ways of completing this formula, but at this point the most obvious guess at how the terms end leads to Bernstein's Schur creation operator (see [Ma] p. 96)*

  $$B_m = \sum_{i \geq 0} (-1)^i s_{(m+i)} s_{(1^i)}^\perp.$$

  Of course, proving that this formula does what we claim it is supposed to is another matter.

- Creation operators for q-analogues often share a curious relationship with their non-q-ified counterparts. Observing that this relationship holds gives us 'for free' formulas for q-creation operators. For $s_\lambda \circ s_\mu^\perp$ we set $\widetilde{s_\lambda \circ s_\mu^\perp} = s_\lambda \circ s_\mu[X(1-q)]^\perp = \sum_{\gamma,\nu} (-q)^{|\nu|} c^\mu_{\nu\gamma} s_\lambda s_\gamma^\perp s_{\nu'}^\perp$.

  Example 5 *If B_m is Bernstein's Schur creation operator in the previous example then*

  $$\widetilde{B_m} = \sum_{i,j \geq 0} (-1)^i q^j s_{(m+i+j)} s_{(1^i)}^\perp s_{(j)}^\perp = \sum_{j \geq 0} q^j B_{m+j} s_{(j)}^\perp$$

  is a creation operator due to Jing [J] (see also [Ma] p. 237-8 and for this particular version, [G]) which creates the modified Hall-Littlewood basis $H_\mu[X;q] = \sum_\lambda K_{\lambda\mu}(q) s_\lambda$.

Example 6 *[Z2] If B_m is a creation operator for $H_{\mu'}[X;t]$ (surprisingly, any one will will work) then $\widetilde{B_m}$ is a creation operator for the modified Macdonald basis $H_\mu[X;q,t] = \sum_\lambda K_{\lambda\mu}(q,t)s_\lambda$.*

Remark: The first question everyone always asks me is "why can't you just take the q-twisting of the creation operator for the Schur basis and then do it again in the parameter t?" There is no reason you can't...it's just that the resulting basis won't be Schur positive and is nothing that I recognize.

1 Creation operators for generalized Kostka polynomials

Let

$$B_m = \sum_{i\geq 0}(-1)^i s_{(m+i)} s^\perp_{(1^i)}$$

from Example 4. Define $B_\lambda := B_{\lambda_1} \circ B_{\lambda_2} \circ \cdots \circ B_{\lambda_{\ell(\lambda)}}$. It is possible to show that

$$B_\lambda = \sum_\mu \sum_{\gamma:\ell(\gamma)\leq\ell(\lambda)} (-1)^{|\mu|} c^\gamma_{\lambda\mu} s_\gamma \circ s^\perp_{\mu'}.$$

The q-ified version of this operator is

$$\widetilde{B_\lambda} = \sum_\mu \sum_{\gamma:\ell(\gamma)\leq\ell(\lambda)} c^\gamma_{\lambda\mu} s_\gamma \circ s_\mu[(q-1)X]^\perp.$$

The operators $\widetilde{B_\lambda}$ create a family of symmetric functions indexed by sequences of partitions:

$$H_{(\mu^{(1)},\mu^{(2)}\cdots\mu^{(k)})}[X;q] := \widetilde{B_{\mu^{(1)}}}\,\widetilde{B_{\mu^{(2)}}}\cdots\widetilde{B_{\mu^{(k)}}}(1)$$

and we define the generalized Kostka polynomial $K_{\lambda;(\mu^{(1)},\mu^{(2)}\cdots\mu^{(k)})}(q)$ to be the coefficient of s_λ in this symmetric function.

Some things that are known about this family of symmetric functions:

- When $q = 1$ the function specializes to the product,

  $$H_{(\mu^{(1)},\mu^{(2)}\cdots\mu^{(k)})}[X;1] = s_{\mu^{(1)}} s_{\mu^{(2)}} \cdots s_{\mu^{(k)}}.$$

  When $q = 0$,

  $$H_{(\mu^{(1)},\mu^{(2)}\cdots\mu^{(k)})}[X;0] = B_{\mu^{(1)}} B_{\mu^{(2)}} \cdots B_{\mu^{(k)}}(1)$$

  which is either 0 or $\pm$ a single Schur function.

- It is clear from this definition that when the $\mu^{(i)}$ are a weakly decreasing sequence of one row partitions then the $\widetilde{B_{\mu^{(i)}}}$ are equal to Jing's creation operators and hence $H_{(\mu^{(1)},\mu^{(2)},\cdots,\mu^{(k)})}[X;q]$ is just the Hall-Littlewood polynomial we called $H_\mu[X;q]$ in Example 5.

- In this definition, the symmetric function very much depends on the order of the sequence of partitions (i.e. except in very rare circumstances, $H_{(\mu^{(1)},\mu^{(2)})}[X;q] \neq H_{(\mu^{(2)},\mu^{(1)})}[X;q]$). This dependence on order of the partitions does not exist for the definitions of 'generalized Kostka polynomials' which come from representation theory or rigged configurations (but these definitions only currently exists for a sequence of rectangles and not for a general sequence of partitions).

- When each of the $\mu^{(i)}$ are rectangles and the widths of these rectangles are weakly decreasing, it has been shown that $K_{\lambda;(\mu^{(1)},\mu^{(2)},\cdots,\mu^{(k)})}(q)$ is a generating function for tensors of crystals (combine [KSS], [Sh], [SZ] and possibly other references) :

$$K_{\lambda;(\mu^{(1)},\mu^{(2)},\cdots,\mu^{(k)})}(q) = \sum_{T \text{ highest weight}} q^{coenergy(T)}$$

Some things that are **not** known about this family of symmetric functions:

- If the sequence of partitions $(\mu^{(1)},\mu^{(2)},\cdots,\mu^{(k)})$ concatenates to be a partition (that is, if the last part of $\mu^{(i)}$ is greater than or equal to the first part of $\mu^{(i+1)}$ for all i), then $H_{(\mu^{(1)},\mu^{(2)},\cdots,\mu^{(k)})}[X;q]$ is conjectured to be Schur positive.

- If each of the $\mu^{(i)}$ are rectangles and the widths of these rectangles are weakly decreasing, then $q^{\sum_i (i-1)|\mu^{(i)}|} H_{(\mu^{(1)},\mu^{(2)},\cdots,\mu^{(k)})}[X;q^{-1}]$ is conjectured to be equal to the LLT symmetric function whose quotient is indexed by the same sequence of rectangles.

1.1 Maple programs for computing 'generalized Kostka polynomials'

```
# Load the SF package
with(SF);

# computes s_λ s_μ but with the restriction that no partition in
# the result has length longer than length(λ)
restrmult:=proc(la,mu) local sEX, sids, v;
sEX:=tos(s[op(la)]*s[op(mu)]);
sids:=select(x->evalb(nops(x)>nops(la)),
select(x->evalb(op(0,x)=‘s‘),indets(sEX,‘indexed‘)));
subs({seq(v=0,v=sids)},sEX);
end:

# The creation operator B̃_μ which builds the
# generalized Hall-Littlewood symmetric functions.
```

```
# remark:  theta(theta(P[X],q,0),-1) = P[(q − 1)X]
Bmutilde:=proc(mu,EX) local n, ga;
tos(s[op(mu)]*EX+add(add(restrmult(mu,ga)*skew(
theta(theta(s[op(ga)],q,0),-1),EX), ga=Par(n,nops(mu))),
n=1..stdeg(EX)));
end:

# The generalized Hall-Littlewood functions H_(mu^(1),mu^(2),...,μ^(k))[X;q]
H:=proc(partlst) local out,i;
out:=s[op(partlst[-1])];
for i from 2 to nops(partlst) do
out:=Bmutilde(partlst[-i],out);
od;
out;
end:
```

Example:

```
> Bmutilde([2,1],s[1]);
```

$$qs_{3,1} + qs_{2,2} + s_{2,1,1}$$

```
> H([[3,2],[2,1],[1]]);
```

$$q^5 s_{6,3} + q^4 s_{6,2,1} + 2\,q^5 s_{5,4} + q^3\,(3\,q+1)\,s_{5,3,1} + 2\,q^3 s_{5,2,2} + q^2\,(1+q)\,s_{5,2,1,1} + q^3\,(2\,q+1)\,s_{4,4,1} + q^2\,(3\,q+1)\,s_{4,3,2} + 2\,q^2\,(1+q)\,s_{4,3,1,1} + q\,(2\,q+1)\,s_{4,2,2,1} + qs_{4,2,1,1,1} + q^2 s_{3,3,3} + q\,(2\,q+1)\,s_{3,3,2,1} + qs_{3,3,1,1,1} + qs_{3,2,2,2} + s_{3,2,2,1,1}$$

1.2 MuPAD programs for computing 'generalized Kostka polynomials'

```
# makes an alias to the domain of symmetric functions called 'S'
S:=examples::SymmetricFunctions(Dom::ExpressionFieldWithDegreeOneElements([t,q])
,vHL=t,vMcd=q);

# the plethystic substitution P[X] -> P[(q-1)X]
Vq:=EX->map(S::s(S::plethysm(EX,(q-1)*S::p([1]))),simplify);

# kills Schur functions which are indexed by a partition with length ≥ k
killtoohigh:=(EX,k)->_plus(S::s::monomial(op(u)) $ u in
select(poly2list(S::s(EX)),(x,k)->is(nops(op(x,2))<=k),k));

# skew( s_λ, s_μ ) = s_μ/λ and it is bilinear
skew:=operators::makeBilinear(prog::bless(
```

```
(part1,part2)->S::s([part2,part1]), S), Source1 = S::s, Source2 = S::s,
ImageSet = S::s );

# this finds the degree of the symmetric function
stdeg:=EX->S::s::degree(S::s(EX));
```

\# The creation operator $\widetilde{B_\mu}$ which builds the

```
# generalized Hall-Littlewood symmetric functions.
Bmutilde:=proc(mu,EX) local n, ga;
begin
map(S::s(S::s(mu)*EX +
_plus(_plus(killtoohigh(S::s(mu)*S::s(ga),nops(mu))*skew(Vq(S::s(ga)),EX)
$ ga in combinat::partitions::list(n, MaxLength=nops(mu)))
$ n = 1..stdeg(EX))), simplify);
end_proc;
```

\# The generalized Hall-Littlewood functions $H_{(mu^{(1)},mu^{(2)},\cdots,\mu^{(k)})}[X;q]$

```
H:=proc(partlst) local out,i;
begin
out:=S::s(partlst[-1]);
for i from 2 to nops(partlst) do
out:=Bmutilde(partlst[-i], out);
end_for;
out;
end_proc;
```

Example:

```
>> Bmutilde([2,1], S::s([1]));
```

$$qs_{3,1} + s_{2,1,1} + qs_{2,2}$$

```
>> H([[3,2],[2,1],[1]]);
```

$$s_{3,2,2,1,1} + qs_{3,2,2,2} + qs_{3,3,1,1,1} + q\ (2\,q+1)\,s_{3,3,2,1} + q^2 s_{3,3,3} + qs_{4,2,1,1,1}$$
$$+2\,q^2\ (q+1)\,s_{4,3,1,1} + q^2\ (3\,q+1)\,s_{4,3,2} + q^3\ (2\,q+1)\,s_{4,4,1} + q^2\ (q+1)\,s_{5,2,1,1}$$
$$+2\,q^3 s_{5,2,2} + q^3\ (3\,q+1)\,s_{5,3,1} + 2\,q^5 s_{5,4} + q^4 s_{6,2,1} + q^5 s_{6,3} + q\ (2\,q+1)\,s_{4,2,2,1}$$

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