PROBLEMS RELATED TO MACDONALD POLYNOMIALS

This document is essentially a transcript of remarks from the first open problem session of the Kostka workshop, which was moderated by Viswanath Sankaran. The transcriber (Nick Loehr) takes full responsibility for any errors and garbles appearing below. The following problems all involve aspects of Macdonald polynomials and their generalizations.

(1) Prove Schur positivity of $\tilde{H}_\mu$, perhaps by using Haglund's new combinatorial formula for the monomial expansion of modified Macdonald polynomials.

(2) Give a combinatorial proof of the identity

$$\tilde{H}_\mu(x;q,t) = \tilde{H}_{\mu'}(x;t,q).$$

More explicitly, find a bijection $T \mapsto T'$ from the set of fillings of $\mathrm{dg}(\mu)$ to the set of fillings of $\mathrm{dg}(\mu')$ such that $x^T = x^{T'}$, $\mathrm{inv}(T) = \mathrm{maj}(T')$, and $\mathrm{maj}(T) = \mathrm{inv}(T')$.

(3) Give a reasonable definition for *modified* Macdonald polynomials for other root systems besides A_n. We remark that much of the audience reacted to the possibility of such a generalization with considerable skepticism. One immediate problem is determining what the analogue of plethysm should be (recall that the axioms defining the usual modified Macdonald polynomials involve the plethystic transformations $\tilde{H}_\mu[X(1-q)]$ and $\tilde{H}_\mu[X(1-t)]$). John Stembridge raised the question of what indexing set should be used for the purported generalizations (partitions may no longer work). He also suggested that there may be a bifurcation (in some sense) between representation theory of the Weyl group and representation theory of the Lie algebra once we leave type A, which would need to be dealt with somehow. Others suggested using Haglund's combinatorial definition as a starting point for seeking analogues in other root systems. This approach may ultimately be more fruitful than the algebraic or representation-theoretical approaches, but it also seems quite difficult.

We should point out that the theory of "ordinary" Macdonald polynomials (P_μ and the integral form J_μ) has already been extended in a very satisfactory way to general root systems. See Macdonald's "other" book, *Affine Hecke Algebras and Orthogonal Polynomials*, for a complete account.

(4) Another potential generalization of the modified Macdonald polynomials $\tilde{H}_\mu$ involves replacing the indexing partition μ by an arbitrary diagram $S \subseteq \mathbb{N} \times \mathbb{N}$. For diagrams S satisfying suitable conditions, $\tilde{H}_S$ can be defined (conjecturally) by solving systems of recurrences that express $\frac{\partial}{\partial p_1}\tilde{H}_S$ (and similar quantities) as explicit combinations of $\tilde{H}$'s indexed by diagrams with fewer squares. Rather than attempting to recreate all the details of this description, we merely refer the reader to the following thorough (91 page) reference: F. Bergeron, N. Bergeron, A. Garsia, M. Haiman, and G. Tesler, "Lattice diagram polynomials and extended Pieri rules," *Adv. in Math.* **142** (1999),

244—334. A warning was issued at the workshop that Conjecture I.1 in this reference is not true in the stated generality. Nevertheless, Conjecture I.3 (which is the main conjecture of the paper) is still open. The same is true for all the other conjectures of the paper.

Garsia raised the problem of determining generalizations of Haglund's statistics that would give combinatorial versions of the general polynomials $\tilde{H}_S$. He asserted that in some special cases (like two-row "pistol" shapes), the obvious analogues of Haglund's statistics already work. But there exist other simple diagrams (like "BOZO") where the combinatorial construction seems to fail.

Mark Haiman sketched a more algebraic (conjectural) description of the polynomials $\tilde{H}_S$. Label the rows and columns of the diagram of S with arbitrary distinct complex numbers. Say $|S| = n$. Writing down the row and column labels of each cell in S (in some order) gives a certain element of $\mathbb{C}^{2n}$. Consider the S_n-orbit of this element (corresponding to the elements in $\mathbb{C}^{2n}$ obtained by visiting the cells of S in other orders); let $I \subset R = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be the ideal of this orbit. The quotient ring R/I has two filtrations. For the first filtration, let $f_x^{(d)}$ be the image in R/I of polynomials in R of degree at most d in the x-variables. The second filtration $f_y^{(d)}$ is defined similarly. Introduce a bigrading of R/I by letting the i, j-bigraded piece be

$$\frac{f_x^{(i)} \cap f_y^{(j)}}{f_x^{(i-1)} \cap f_y^{(j)} + f_x^{(i)} \cap f_y^{(j-1)}}.$$

We thereby obtain a doubly graded ring carrying the left regular representation of S_n — *but* this ring need not be a quotient ring of R! It is conjectured that the Frobenius series of this bigraded ring is precisely $\tilde{H}_S$. Another conjecture is that nothing depends on the complex numbers initially chosen to index the rows and columns of S. Remark: If S is the diagram of a partition, the fact that the bigraded ring above is a quotient ring of R is equivalent to the $n!$ theorem. There is further structure when S is obtained from a partition by removing a single cell. Again we refer the reader to the paper mentioned above for full details.