ARTICLES ON MACDONALD POLYNOMIALS

The following references discuss various aspects of Macdonald polynomials. They appear in chronological, rather than alphabetical order.

(1) Ian Macdonald, *Symmetric Functions and Hall Polynomials* (second edition). Oxford University Press, 1995.

This book is the standard encylopedic reference for symmetric functions, Hall-Littlewood polynomials, and Macdonald polynomials. Macdonald's viewpoint is highly algebraic, definitely NOT combinatorial, which may make the book tough reading for more combinatorial-minded readers. A huge amount of information from the literature is presented in the form of long exercises at the end of each section (called "examples" by Macdonald).

(2) Garsia and Haiman, "A graded representation model for Macdonald's polynomials." *Proc. Nat. Acad. Sci. USA* **90** (1993), #8, 3607—3610.

This paper announces the $n!$ conjecture and the conjectured combinatorial-representation theoretic interpretation of the Macdonald q,t-Kostka coefficients. The paper sketches some results supporting the conjectures.

(3) Mark Haiman, "Conjectures on the quotient ring by diagonal invariants," *J. Algebraic Combin.* **3** (1994), #1, 17—76.

This paper gives combinatorial conjectures and supporting theorems concerning the ring $R_n = \mathbb{Q}[x_1, y_1, \ldots, x_n, y_n]/I$, where I is the ideal generated by S_n-invariants. The paper contains tables of the Hilbert series of R_n for $n \leq 7$ and the Frobenius series for $n \leq 6$.

(4) Garsia and Haiman, "Some natural bigraded S_n-modules and q,t-Kostka coefficients," *Foata Festschrift. Electron. J. Combin.* **3** (1996), #2, R24, 60 pp.

This paper gives detailed proofs of results announced in the PNAS paper (above) and conducts a further study of the modules introduced there.

(5) Garsia and Haiman, "A remarkable q,t-Catalan sequence and q-Lagrange inversion," *J. Algebraic Combin.* **5** (1996), #3, 191—244.

The paper contains a conjectured exact formula, in terms of Macdonald polynomials, for the Frobenius series of the diagonal harmonics. Theorem: various specializations of the master formula imply all the combinatorial conjectures in the earlier JACO paper (above).

(6) Garsia and Haiman, "A random q,t-hook walk and a sum of Pieri coefficients," *JCTA* **82** (1998), #1, 74—111.

This paper modifies the classical hook-walk process of Greene, Nijenhuis, and Wilf by weighting the transition probabilities by powers of indeterminates q and t. This simple modification leads to identities involving the Pieri coefficients for Macdonald polynomials which generalize familiar enumerative identities for standard Young tableaux.

(7) Brockman and Haiman, "Nilpotent orbit varieties and the atomic decomposition of the q-Kostka polynomials," *Canad. J. Math.* **50** (1998), #3, 525—537.
Theorem: There exist non-negative polynomials $R_{\lambda,\mu}(q)$ such that the q-Kostka polynomial $K_{\lambda,\mu}(q)$ is the sum of $R_{\lambda,\nu}(q)$ for all $\nu \geq \mu$ in dominance order. The proof is by geometry and representation theory. The result was previously known from Lascoux and Schützenberger's combinatorial theory of cyclage and "atoms."

(8) Mark Haiman, "t, q-Catalan numbers and the Hilbert scheme," *Discrete Math.* **193** (1998), 201—224.
The paper gives a geometric interpretation of the t, q-Catalan number rational function formula using the Hilbert scheme of points in the plane. Theorem: the formula reduces to a polynomial in q and t. The "higher" Catalan polynomials $C^m(q,t)$ have nonnegative coefficients for m sufficiently large.

(9) Mark Haiman, "Macdonald polynomials and geometry" in *New Perspectives in Geometric Combinatorics, MSRI Publications* **37** (1999), 207—254.
This paper explicates the connection between Macdonald polynomials, the $n!$ conjecture, and the Hilbert scheme of points in the plane and related algebraic varieties. Theorem: $n!$ conjecture implies Macdonald positivity conjecture. Additional results on diagonal harmonics may also be found here.

(10) F. Bergeron, N. Bergeron, Garsia, Haiman, and Tesler, "Lattice diagram polynomials and extended Pieri rules," *Adv. in Math.* **142** (1999), 244—334.
Analog of the $n!$ conjecture with Young diagrams replaced by general subsets $D \subseteq \mathbb{N} \times \mathbb{N}$. When D is a Young diagram with a hole missing, we conjecture character formulas which are connected with the Pieri formulas for Macdonald polynomials.

(11) F. Bergeron, Garsia, Haiman, and Tesler, "Identities and positivity conjectures for some remarkable operators in the theory of symmetric functions," *Methods and Applications of Analysis* **6**, #3 (1999), 363—420.
Continued study of the nabla operator which is central to plethystic formulas for Macdonald polynomials, the conjectured character formula for diagonal harmonics, and related matters. Theorems: nabla is a polynomial operator; the conjectured character formula for diagonal harmonics is a polynomial. Positivity conjectures for a large class of related formulas are also given.

(12) Garsia, Haiman, and Tesler, "Explicit plethystic formulas for Macdonald q,t-Kostka coefficients," *The Andrews Festschrift. Seminaire Lotharingien* **42** (1999), electronic, 45 pp.

A simple and explicit formula for the transformed Macdonald polynomial is given using the nabla operator which has come to play a central role in the theory. From this we obtain new and simple proofs of plethystic formulas and integrality for q,t-Kostka coefficients, along with other results such as Sahi's interpolation theorem and Macdonald-Koornwinder reciprocity.

(13) Mark Haiman, "Hilbert schemes, polygraphs, and the Macdonald positivity conjecture," *J. Amer. Math. Soc.* **14** (2001), 941—1006.

The isospectral Hilbert scheme of points in the plane is shown to be normal, Cohen-Macaulay, and Gorenstein. This implies the $n!$ conjecture and the positivity conjecture for the Kostka-Macdonald coefficients. It also implies that the Hilbert scheme of points in the plane coincides with the Hilbert scheme of regular S_n-orbits in $\mathbb{C}^{2n}$.

(14) Mark Haiman, "Vanishing theorems and character formulas for the Hilbert scheme of points in the plane," *Invent. Math.* **149**, #2 (2002), 371—407.

We derive the character formula for diagonal harmonics from vanishing theorems for tautological bundles on the Hilbert scheme of points in the plane. In particular this implies that the space of diagonal harmonics has dimension $(n+1)^{n-1}$. The vanishing theorems are proved using results from the preceding paper together with a recent theorem of Bridgeland, King, and Reid.

(15) Mark Haiman, "Notes on Macdonald polynomials and the geometry of Hilbert Schemes," in *Symmetric Functions 2001: Surveys of Developments and Perspectives, Proceedings of the NATO Advanced Study Institute 2001*, ed. Sergey Fomin. Kluwer, Dordrecht (2002) 1—64.

Notes from a series of lectures given in the combinatorics seminar at UCSD, Spring 2001, giving a user-friendly introduction to results in the preceding two papers. They also contain information on the combinatorial description of the basis of the polygraph ring.

(16) Mark Haiman, "Combinatorics, symmetric functions, and Hilbert schemes," in *CDM 2002* (2003), 39—112.

Survey article on the proof of the Macdonald positivity, $n!$, and $(n+1)^{n-1}$ conjectures using new results on the geometry of Hilbert schemes. Includes background material from combinatorics, symmetric function theory, representation theory and geometry, and discussion at the end of future directions, new conjectures, and related work of Ginzburg, Kumar and Thomsen, Gordon, and Haglund and Loehr.

(17) Haglund, Haiman, Loehr, Remmel, and Ulyanov, "A combinatorial formula for the character of the diagonal coinvariants," *Duke Math. J.* **126** (2005), 195—232.

This paper conjectures a combinatorial interpretation for $\nabla(e_n)$, which is the Frobenius series of the module of diagonal coinvariants for the symmetric group. The combinatorial formula is a sum of labelled Dyck paths weighted by suitable statistics. The formula can also be rewritten as a weighted sum of LLT polynomials indexed by shifted column shapes. By appealing to Kazhdan-Lusztig theory, the paper proves the symmetry and Schur-positivity of the combinatorial formula. Similar conjectures and results are given for the higher powers $\nabla^m(e_n)$.

(18) Jim Haglund, "A combinatorial model for the Macdonald polynomials," *Proc. Natl. Acad. Sci. USA* **101** (2004), 16127—16131.

This article conjectures a combinatorial interpretation for (modified) Macdonald polynomials, which was subsequently proved by Haglund, Haiman, and Loehr. Haglund's new combinatorial definition expresses the Macdonald polynomial $\tilde{H}_\mu$ as a sum of monomials constructed from fillings of the diagram of μ that are weighted by two new statistics. The paper also defines more general combinatorial polynomials indexed by arbitrary diagrams in the first quadrant. These latter polynomials are shown to be symmetric functions, and a recursion is given characterizing these polynomials. Finally, Haglund conjectures a formula for the Kostka-Macdonald coefficients when μ has at most three columns.

(19) Haglund, Haiman, and Loehr, "A combinatorial formula for Macdonald polynomials," *J. Amer. Math. Soc.* **18** (2005), 735—761.

This paper proves Haglund's conjectured combinatorial model for the modified Macdonald polynomials. The proof gives a combinatorial interpretation for the algebraic axioms characterizing Macdonald polynomials and then verifies the axioms using two sign-reversing involutions. Other results found in this paper include: description of Macdonald polynomials in terms of LLT polynomials; a simple combinatorial proof of the Lascoux-Schützenberger charge formula for Hall-Littlewood polynomials; an extension of the Knop-Sahi formula for Jack polynomials to integral Macdonald polynomials; a new description of the Kostka-Macdonald coefficients for partitions with at most two columns, which is proved using crystals; and an elementary combinatorial proof that LLT polynomials are symmetric functions.