Introduction to LLT-polynomials

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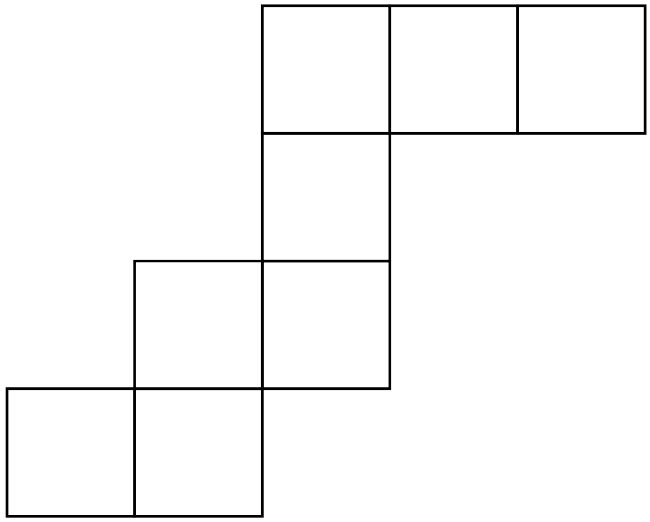
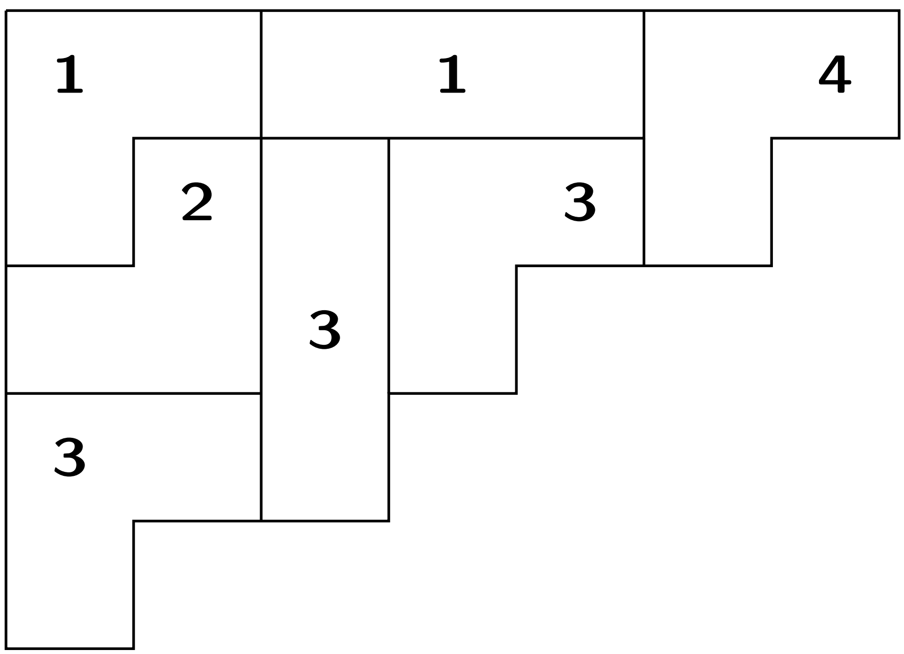
$R =$



A ribbon R with 8 boxes, 4 rows and $\text{spin}(R) = 4 - 1 = 3$.



A semistandard 3-ribbon tableau with shape $(7, 6, 4, 3, 1)$, weight $(2, 1, 3, 1)$ and spin 7. The spin of a ribbon tableau is the sum of the spins of its component ribbons.

Ribbon Functions. Lascoux, Leclerc and Thibon's ribbon tableaux generating functions (LLT polynomials):

$$\mathcal{G}^{(n)}_{\lambda/\mu}(X;q) = \sum_{T} q^{\mathsf{spin}(T)} \mathbf{x}^T$$

where the sum is over semistandard n-ribbon tableaux of shape λ.

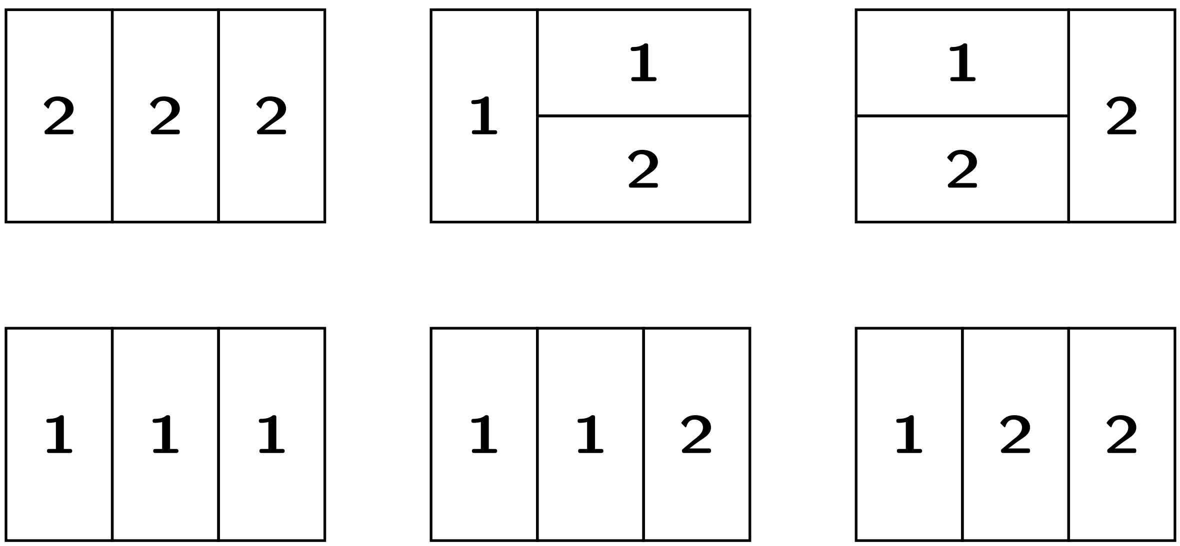
1. $\mathcal{G}^{(n)}_{\lambda/\mu}(X;q)$ is symmetric.

2. $\mathcal{G}^{(n)}_{\lambda/\mu}(X;1)$ is a product of Schur functions.

3. Write $\mathcal{G}^{(n)}_{\lambda}(X;q) = \sum_{\mu} c^{\mu}_{\lambda}(q) s_{\lambda}(X)$. Then $c^{\mu}_{\lambda}(q) \in \mathbb{N}[q]$.

Example. Let $n = 2$ and $\lambda = (3, 3)$. Then we have

$$\mathcal{G}^{(2)}_{(3,3)}(x_1, x_2; q) =$$
$$q^3(x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3) + q(x_1^2 x_2 + x_2^2 x_1).$$



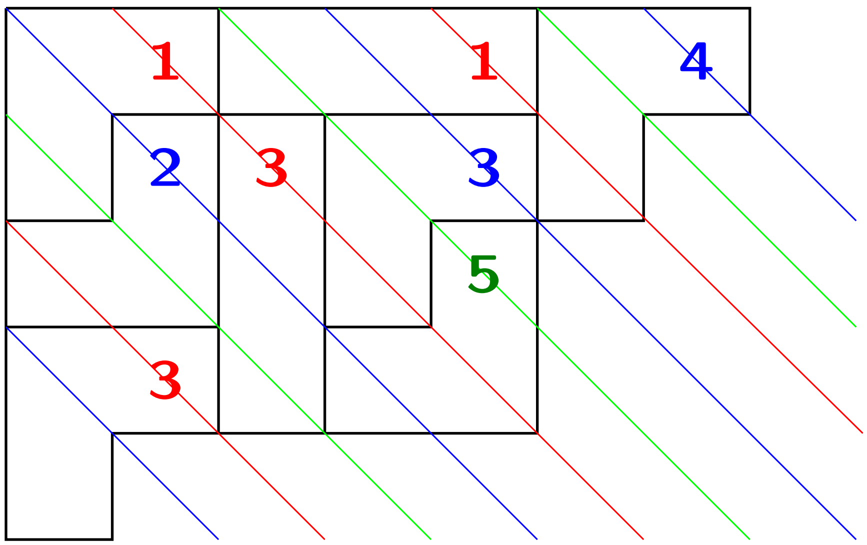
In fact,

$$\mathcal{G}^{(2)}_{(3,3)}(X; q) = q s_{2,1}(X) + q^3 s_3(X).$$

Note that the symmetry is already quite non-obvious. Note also that

$$\mathcal{G}^{(2)}_{(3,3)}(X; 1) = s_1 s_2.$$

n-quotient bijection.



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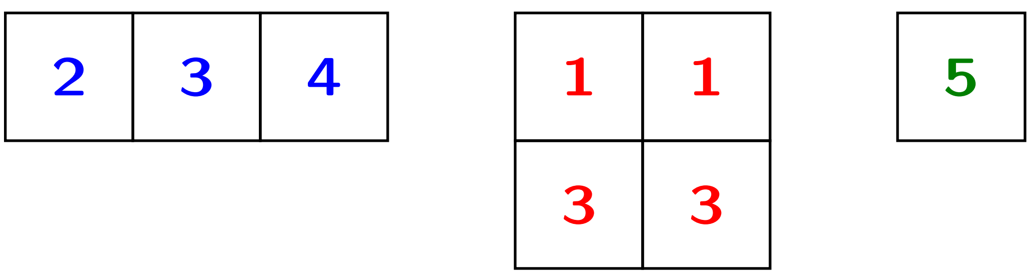

Some Known Things.

- The functions $\mathcal{G}_\lambda^{(n)}(X;q)$ generalise the Hall-Littlewood polynomials (LLT).

- There are "ribbon Pieri" and "ribbon Cauchy" formulae (Lam).

- There is a "ribbon insertion" algorithm, which behaves like Schensted insertion and preserves the spin (van Leeuwen).

Some Unknown Things.

- There is no closed formula (determinantal, Jacobi-Trudi) for $\mathcal{G}_{\lambda}^{(n)}(X;q)$ or $\mathcal{G}_{\lambda/\mu}^{(n)}(X;q)$.

- The general Schur positivity of the skew functions $\mathcal{G}_{\lambda/\mu}^{(n)}(X;q)$ is not known except for $n = 2$.

- How they multiply (ribbon Littlewood-Richardson rule) is unknown.