GENERALIZED KOSTKA POLYNOMIALS AS PARABOLIC LUSZTIG q-ANALOGUES

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The (single variable) generalized Kostka polynomials are indexed by a partition λ and a sequence of partitions $R = (R_1, R_2, \dots, R_k)$. Specialized at 1 they yield the Littlewood-Richardson coefficient $\langle s_\lambda \, , \, s_{R_1} s_{R_2} \cdots s_{R_k} \rangle$.

There are essentially three kinds of generalized Kostka polynomials in type A. The first kind is the one-dimensional sum [4, 5], which is a tensor product multiplicity having two conjecturally equal definitions, one (denoted $X_{\lambda;R}(q)$) using the combinatorics of an affine crystal graph and the other by a fermionic formula $M_{\lambda;R}(q)$. These are defined only when R is a sequence of rectangles, but the constructions extend to any affine algebra. X and M are known to be equal for type A [7], for tensor products of single rows in unexceptional affine type [19], and for single columns in type D [15]. The second kind is the Lascoux-Leclerc-Thibon polynomials $c_{\lambda;R}(q)$, which come from the action of a Heisenberg algebra on deformed Fock space and are defined using ribbon tableaux [9] [12]. These polynomials are known to have nonnegative integer coefficients for any sequence of partitions R [8]. They also appear prominently in formulae for Macdonald polynomials [2] and diagonal coinvariants [3]. The third kind is a (virtual) graded multiplicity $K_{\lambda;R}(q)$ in a twisted module supported in the closure of the conjugacy class of a nilpotent matrix [20]. These may be regarded as a parabolic analogue of Lusztig's q-analogue of weight multiplicity [13]. The polynomials $K_{\lambda;R}(q)$ can also be defined using a parabolic analogue [21] of Jing's Hall-Littlewood creation operators [1] [6]. They also appear in a definition of the graded k-Schur function [10], which arose in the study of Macdonald polynomials.

Our goal is to give a quick definition of the third kind of generalized Kostka polynomial. This follows [20].

Fix a partition λ and any sequence of partitions $R = (R_1, R_2, \dots, R_k)$. These given, we make a number of definitions. Let $\eta = \eta(R) = (\eta_1, \eta_2, \dots, \eta_k)$ where η_i is the number of parts in the partition R_i and let $N = \sum_i \eta_i$. Let $\gamma = \gamma(R) \in \mathbb{Z}^N$ be the sequence of integers given by the parts of R_1, followed by the parts of R_2, etc. Let $R^+_\eta \subset \{1, 2, \dots, N\}^2$ be the subset of positions in an $N \times N$ matrix, that are strictly above the block diagonal whose diagonal blocks have sizes $\eta_1, \eta_2, \dots$.

Example 1. Let $R = ((4,4),(3),(1,1,1))$. Then $\eta = (2,1,3)$, $N = 6$, $\gamma = (4,4,3,1,1,1)$, and R^+_η is given by the positions marked with an x.

$$\begin{matrix} d & d & x & x & x & x \\ d & d & x & x & x & x \\ . & . & d & x & x & x \\ . & . & . & d & d & d \\ . & . & . & d & d & d \\ . & . & . & d & d & d \end{matrix}$$

So $R_\eta^+ = \{(1,3),(1,4),(1,5),(1,6),(2,3),(2,4),(2,5),(2,6),(3,4),(3,5),(3,6)\}$.

Let $P_\eta(\beta) \in \mathbb{Z}_{\geq 0}[q]$ be the polynomial defined by the generating series

$$\sum_{\beta \in \mathbb{Z}^n} x^\beta P_\eta(\beta) = \prod_{(i,j) \in R_\eta^+} \frac{1}{1 - q\frac{x_i}{x_j}}$$

where we regard the right hand side as a product of geometric series. Let $\rho = (N-1, N-2, \dots, 1, 0) \in \mathbb{Z}^N$. The generalized Kostka polynomial $K_{\lambda;R}(q) \in \mathbb{Z}[q]$ is defined by the alternating sum over the symmetric group

$$K_{\lambda;R}(q) = \sum_{w \in S_N} (-1)^{\ell(w)} P_\eta(w(\lambda+\rho) - (\gamma+\rho)). \tag{1}$$

These polynomials can be computed quite efficiently using a recurrence [20] that generalizes the Morris recurrence [14] for Kostka-Foulkes polynomials.

If $\mu = (\mu_1, \dots, \mu_k)$ is a partition and $R_i = (\mu_i)$ is a single-part partition for each i then $K_{\lambda;R}(q) = K_{\lambda;\mu}(q)$ is the Kostka-Foulkes polynomial, which is Lusztig's q-analogue of weight multiplicity.

Conjecture 2. *(B. Broer) If $\gamma = \gamma(R)$ is a partition then $K_{\lambda;R}(q)$ has nonnegative integer coefficients.*

Under these hypotheses, a conjectural combinatorial formula was given in [20] using catabolizable tableaux with the usual charge statistic.

Lascoux and Schützenberger realized the Kostka-Foulkes polynomials as the generating function over semistandard tableaux with the charge statistic [11]. Simultaneously and independently, in the special case where each of the partitions R_i is a rectangle, [16] and [18] used a kind of Littlewood-Richardson tableau and a generalization of the charge statistic, to give a combinatorial generalization of the combinatorics of Lascoux and Schützenberger to the rectangle case. These combinatorially-defined polynomials were shown to coincide with the one-dimensional sums $X_{\lambda;R}(q)$ in [16] and [17]. In this combinatorics the order of the rectangles in R is immaterial. In [18] the LR tableau definition was shown to coincide with the definition in (1).

Theorem 3. [18] *If R_i is a rectangular partition for all i and $\gamma = \gamma(R)$ is a partition then $K_{\lambda;R}(q)$ has nonnegative coefficients with an explicit combinatorial description.*

It is conjectured that in the rectangle case, the LLT polynomials agree with the other two kinds of generalized Kostka polynomials.

Conjecture 4. *When R is a sequence of rectangles, $c_{\lambda;R}(q) = K_{\lambda;R}(q)$.*

This is known when all rectangles are single rows or all single columns [9].

We conclude by recounting an important alternative normalization. The cocharge or coenergy generalized Kostka polynomial is defined by

$$\overline{K}_{\lambda;R}(q) = q^{||R||} K_{\lambda;R}(q^{-1}) \tag{2}$$

where

$$||R|| = \sum_{1 \leq i < j \leq k} |R_i \cap R_j| \tag{3}$$

is the sum of the sizes of the partitions given by intersecting the partition diagrams of all pairs of partitions in R.

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