THE $K = LLT$ CONJECTURE

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Let $R = (R_1, R_2, \dots, R_k)$ be a fixed sequence of rectangular partitions. The generating functions $\overline{H}_R(x;q)$ and $\overline{G}_R(x;q)$ and the coefficients $\overline{K}_{\lambda;R}(q)$ (the K- or coenergy generalized Kostka polynomials) and $\overline{c}_{\lambda;R}(q)$ (the cospin LLT polynomials) are defined by

$$\overline{H}_R(x;q) = \sum_T q^{\mathrm{coenergy}(T)} x^T = \sum_\lambda \overline{K}_{\lambda;R}(q)\, s_\lambda(x)$$
$$\overline{G}_R(x;q) = \sum_T q^{\mathrm{cospin}(T)} x^T = \sum_\lambda \overline{c}_{\lambda;R}(q)\, s_\lambda(x)$$

where $T = (T_1, T_2, \dots, T_k)$ runs over tableau lists such that T_i is a semistandard tableau of shape R_i for all i, x^T is the monomial whose exponent is the total content of the tableau list T, and coenergy and cospin are statistics on tableau lists whose definitions can be found in Shimozono's and Lam's lecture notes respectively.

The generating functions $\overline{H}_R(x;q)$ and $\overline{G}_R(x;q)$ are known to be symmetric so that the coefficients $\overline{K}_{\lambda;R}(q)$ and $\overline{c}_{\lambda;R}(q)$ are well-defined. It is known that both the K polynomials $\overline{K}_{\lambda;R}(q)$ and the LLT polynomials $\overline{c}_{\lambda;R}(q)$ have nonnegative integer coefficients. It is clear from the tableau definition of a Schur function that

$$\overline{c}_{\lambda;R}(1) = \overline{K}_{\lambda;R}(1) = \langle s_\lambda \,,\, s_{R_1} \cdots s_{R_k} \rangle$$

is the Littlewood-Richardson coefficient.

The $K = LLT$ conjecture asserts that

$$\overline{K}_{\lambda;R}(q) = \overline{c}_{\lambda;R}(q)$$

or equivalently that

$$\overline{H}_R(x;q) = \overline{G}_R(x;q).$$

To prove this it suffices to find a bijection from tableau lists to tableau lists that preserves total content and sends coenergy to cospin.

$K = LLT$ is known to hold in the following cases.

(i) In the two-rectangle ($k = 2$) case. The proof uses very special properties of the Barbasch-Vogan-Garfinkle domino insertion algorithm.

(ii) When each R_i is a single row, both sides are the cocharge Kostka polynomials. The proof of the equality in this case is nontrivial and no combinatorial proof is currently known.

(iii) When each R_i is a single column both sides are the (charge) Kostka polynomials. This can be deduced from the previous case using a transpose duality.

(iv) In the further special case when each R_i is a single box, the tableau lists can be identified with words of length k. If we further restrict to standard content, we are dealing with permutations. In this case the coenergy is maj (the major index or sum of descent positions) and the cospin is inv (the number of inversions). Foata's bijection from the set of permutations to itself, sends maj to inv and preserves the descent set of the inverse permutation. So the desired bijection should be a generalization of Foata's bijection.