The Klein project

The American Institute of Mathematics

The following compilation of participant contributions is only intended as a lead-in to the AIM workshop "The Klein project." This material is not for public distribution.
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CHAPTER A: PARTICIPANT CONTRIBUTIONS

A.1 Andradas, Carlos

As mathematician my work has been mostly in Real Algebraic Geometry, and inside it in semialgebraic and semianalytic, that is subsets of $\mathbf{R}^n$ defined by inequalities of polynomials and analytic functions respectively. I have been president of the Spanish Math Society for 6 years that have as one of its objectives the caring for math education. During that time I had many contacts and cooperation with Spanish math teachers associations. Besides that, I have also been very supportive of initiatives devoted to math education in schools and I have participated in some activities as Math Olympiads. In particular I have been the president of the International Jury of the 2008 IMO held in Madrid. I am convinced of the importance of spreading the use and appreciation of mathematics in society and I have written 3 divulgation books, two addressed to children about 12 years old and a third one to adults. Finally as vice-president of my university I have participated actively in the design and launching of the official master for initial education of math secondary and high school teachers.

A piece of mathematics that I like very much to show the power and beauty of mathematics is the RSA public cryptography method and how we use it (without knowing it) in our daily life.

I am now very interested in game theory and its application in life and in particular in evolutionary games and I think that is some that school teachers should know some basic notions.

A.2 Artigue, Michele

My interest in this workshop comes from the fact that I am a member of the team in charge of piloting the Klein project. Initially, I have worked on the possible content of a chapter on functions and analysis (a topic I have been in charge of in the preparation of math students to the national competition for becoming mathematics high school teachers for ten years in the maths department of my university), and offered contributions on this theme at the first Klein Conference in Madeira and at the Castro Urdiales Conference in Spain. I was also involved in one of the first workshops of the Klein project in Portuguese language in Belo Horizonte, Brasil. Currently, I am working with French colleagues from the logics research team in my university and from the logic group that we have established in the IREM (Institute of research in mathematics education)of the same university (a group in which mathematicians specialized in this area, researchers in math education and teachers work together for preparing educational resources and in service training sessions for secondary teachers)on the logic part of the Klein project. Logics was my first area of research in mathematics and recursivity the theme of my PhD. I am also preparing a Klein vignette on Goodstein sequences with Ferdinando Arzarello, another member of the piloting team of the Klein project.

I hope that this workshop will make the Klein project substantially progress, that I will have the opportunity to exchange and discuss about the current state of our reflection about the "logic chapter" and to get new ideas about possible Klein vignettes in that area really insightful for high school teachers. There is no doubt also that I am still interested in all what concerns functions and analysis in the Klein project. Personally, as a mathematician and a researcher in maths education, I have been involved in the qualitative study of differential equations both from standard and non-standard perspectives and also in the way technology has made it possible to introduce this approach to students with limited mathematical background (first year of university courses in France and even last year of secondary school).

A.3 Baldin, Yuriko

Brief description of my background: I got Doctor of Sciences degree with thesis in Differential Geometry, from State University of Campinas (UNICAMP) in Brazil. I have been to University of California at Santa Barbara and State University of New York at Stony Brook as scholar, in the 80's. I am professor of Mathematics at Federal University of So Carlos, Brazil, with over 30 years of teaching and research experience. Lately I have been working with educational issues mainly about Teacher Education of all levels. I have written two books targeting the undergraduate students and school teachers. I have been appointed as ICMI-IMU representative of Brazil in 2008, and currently, I coordinate The Klein Project in Portuguese aiming to contribute to a modernization of teacher's mathematics knowledge in the actual context of 21st century.

a) Separately, I sent some notes I wrote in English in which I expose some first ideas about the Klein project, but one key subject I like to tell my non mathematician friends is the intuitive idea of curvature of the objects related then to the dynamic idea of this concept with the acceleration vector field of a moving object. This very concrete idea leads to the concept of geodesics which in turn open the way to non-euclidean geometry, using very simple examples, not necessarily starting from axiomatic point of view that is quite abstract to layman.

b) One topic of modern math that I really want to be worked out as well as being understood by secondary school teachers is the connection geometry-algebra, for instance, the geometric interpretation of determinant of matrices as well as the geometric meaning of the multiplication of matrices related to the theory of transformation groups. Other very important issue in the modern times of increasing use of technology, in both real world practical applications and as tools to do and understand mathematics, is the importance of discrete matehmatics and the transition from discrete context to thinking mathematics in a continuous media.

A.4 Bass, Hyman

1. As part of the lead-in to the workshop, we ask that you contribute some information that will help with the workshop preparations. Please write a paragraph or two about your mathematical experiences.

I will send an essay, "Mathematics and Teaching," that may be of interest in this regard.

Then also describe:

a) One piece of mathematics that is your favorite example to tell non-mathematical friends about the power, beauty, or applicability of mathematics.

To me, one of the most powerful and beautiful 'pieces' of mathematics is not so much a theorem, but rather the concept of group, and its multiple uses to capture symmetry, and use it to solve diverse problems – solvability of equations (or integrals) by radicals, relation to Platonic solids, applications to crystallography, fundamental role in particle physics, etc.

b) One piece of modern mathematics that you would like to know about (if you are a secondary math teacher), or that you would like people who teach senior secondary math classes to know about (if you are not one).

Related to part a) I would emphasize the roles that symmetry can play to both illuminate and deepen much mathematics of the school curriculum. In geometry, students are often given the impression that symmetry is an inert property of some geometric object, rather than a transformation that leaves that object invariant.

A.5 Black, Ashli

I am a high school mathematics instructor at a suburban public school north of Seattle. I am currently in my 6th year of teaching and have had the opportunity to teach everything from first year algebra through calculus. Trigonometry is my favorite subject to teach because of the way it blends algebra and geometry together and was the first math topic I felt like I really understood when I was a student. Before I was a teacher I received a BA in Mathematics from Whitman College and worked in the mortgage market as an analyst for a short period of time.

One of my favorite classes to teach is designed to expose students with a history of struggling in mathematics to math topics not typically seen in high school. We rotate through special units covering the basics of cryptography, matrices, graph theory, combinatorics, etc, and I having those students get excited about mathematics when they have never felt that way before is very powerful. While the typical topics taught in the standard high school math line are important skills to build, I feel we do a disservice to the students by not spending more time teaching the more applied-based mathematics. When speaking with non-mathematical friends about math, I like to discuss visual topics such as the Seven Bridges of Konigsberg or doodling "celtic" knots (a la Vi Hart's video). People often think of mathematics as harsh/scary and all equations and I like to turn that perspective around to get them thinking of mathematics as puzzles that can be worked out logically and a way to answer questions about patterns that we can see or create by following certain rules.

The modern mathematics I would like to know more about are fractals and complex numbers and how they could be brought into the high school in a way that students could grasp and work with.

A.6 Cohen, Graeme

I was appointed to a lectureship in the University of Technology, Sydney in 1966 and retired as an associate professor in 2002. I lectured in most of the non-specialist undergraduate areas (calculus, discrete mathematics, linear algebra, complex variables, etc.) to students of engineering and science as well as mathematics majors. On two occasions, I gave a fourth year honours course in number theory.

My research interests are in number theory, with 40 or so publications to do with perfect numbers, amicable numbers and harmonic numbers, with some on certain diophantine equations. Later, I developed an interest in applications of mathematics in sport, and I have papers on tennis, cricket, darts, speed cars, rowing and snooker. I have written three texts on undergraduate mathematics, including "A Course in Modern Analysis and its Applications" (Cambridge University Press, 2003). After retirement, I wrote a book on the history of mathematics in Australia and I have other contributions in that area. I am currently the editor of the Bulletin of the Australian Mathematical Society.

It was also after retirement that I was invited to join the Klein Project Design Group. Two of the Klein Vignettes are mine. I often use the second of these, on tandem rigging of racing eights, in speaking to non-mathematical friends of the applicability of mathematics in unusual settings. The first of my Vignettes, on improving the straightforward way of calculating values of the sine function, is relatively modern mathematics which is understandable by senior secondary mathematics students; they will recognise its usefulness in everyday calculations on a pocket calculator.

A.7 Cuoco, Al

Experiences.
My interest in mathematics education, like many things that affect the lives of people my age, is a lingering consequence of the Vietnam war.

I loved mathematics all through high school—among other things, it was a haven from some of the stressful events happening in my family. Most of the mathematics I learned came from working with one teacher, Frank Kelley (I still see him every now and then, and I always call him "Mr. Kelley," even though the ratio of our ages is rapidly approaching 1). Among other things, he taught me calculus in a sort of a tutorial during what I later realized was his free period. We used an old version of Apostol, and we held our sessions in the back of a larger algebra class that was taught by a first year teacher who was clearly having trouble holding her students' attention. Every now and then, Mr. Kelley would excuse himself, walk over to a student who was acting out, pick the student up by his collar (it was almost always a "he"), shake him a bit, and then return to the calculus discussion at the back of the room. In addition to this and more formal courses, he saw to it that I was in his study periods, where he taught me about partial fractions and other algebraic delights.

I also took geometry with another teacher (who showed up at one of my talks a few years ago); it was under her tutelage that I first thought seriously about proofs. I can probably still locate the page in our geometry book that contains the problem that took me a whole day to solve and that was a central ingredient in my decision to go into mathematics.

I didn't realize it for close to a decade, but the teachers at my high school (a nondescript, small town, working class school) made it possible for someone to pursue mathematics precisely because they didn't treat such an interest as something special or unusual. Mr. Kelley gave me a great deal of encouragement and extra time, but he did the same thing with the members of his track team. Mathematics was simply part of the culture, a path for some people to follow, no more or less important than any other path.

Throughout college, I never questioned the fact that I'd go to graduate school in mathematics. I had no idea what I'd do in graduate school (or even why I wanted to go), but I never gave it much thought. This was in the late 60's, so no one thought much about the future; my mind was on opposition to the war and getting married (which we did during my senior year).

I graduated in 1969 and headed for Northeastern to study group theory (Gorenstein was there). That was the year they did away with graduate deferments. There were three choices for me:

- move to Canada

- get a medical deferment
- get a deferment by teaching school

The first choice really wasn't an option—I had barely been out of New England; leaving the country was more work that I could imagine— so I concentrated on the second two: I tried to get unhealthy, and I applied for high school teaching jobs.

This was during the great "teacher shortage." Schools were having a difficult time finding mathematics teachers, and many districts were willing to hire anyone with a degree remotely related to mathematics. I had an uncle in a nearby city who had an "in" with the local school board. I was interviewed and hired in July of 1969 at a starting salary of $6600. I had never taken an education course, never been in a high school mathematics class since my own high school years, and had no idea what I intended to do with my students in September. I simply wanted to wait out the war so we could get back to the original plan of graduate school.

Things got off to a rocky start. I taught geometry from the same book that I used in high school. The geometry class was "low level," something I finally deduced from the fact that several of the students in the class seemed to be about my age. I merrily discussed axioms and postulates until one of the students ran to the department chair and complained that this was her last chance to pass geometry and she didn't want to participate in the breaking in of a new teacher.

But gradually I learned, and gradually I became hooked. I learned to put up with the insane working conditions, *really* insane administrators (our principal used to announce in the spring when the weather was warm enough to warrant men teachers removing their jackets), and schizophrenic demands. I saw that students appreciated teachers who loved the subject they were teaching, and I slowly realized that high school students were capable of doing much more mathematics than adult teachers, curriculum designers, and educational theorists would admit.

I eventually did go to graduate school. The war wound down just as I became too old to be eligible for the draft, so I took a couple years leave to get a Ph.D. (my interests had changed from group theory to number theory, so I ended up at Brandeis). But it was very clear throughout graduate school that I'd return to high school teaching.

When I did return, I continued to do research in number theory for a few years at the same time that my wife and I were building a house. These two experiences— mathematical research and house building—had a profound effect on my high school teaching. For example, it became evident to me that, while the objects involved in mathematical research are out of reach for most high school students, the *methods* used by mathematicians are quite accessible to young people. And the kinds of thinking involved in designing and building something that doesn't yet exist were strikingly similar to the kinds of things that went through my head during my long bouts with power series over local fields.

I began to concentrate on these mathematical ways of thinking in my classes, using them both to introduce students to genuine mathematical experiences in elementary contexts and to illustrate the coherence and simplicity of mathematics itself, things that were completely missing from the awful curricula that were around at the time. This "habits of mind" focus still is the centerpiece of my work in education. So, what started as a short detour into high school teaching has grown into a lifelong fascination with mathematics education, with making mathematical thinking the focus of curriculum, and with pushing the edges of what we expect from young people.

One piece of mathematics for non-mathematical friends..
I like to pose questions that require little or no technical background but that exemplify important mathematical thinking, like:

- Was there a time in your life when your height in inches was the same as your weight in pounds? (This is due to Tom Banchoff.)
- (When) is the average of two averages the average of the whole lot?
- How many clinks are there if everyone at the table toasts everyone else?

Mathematics for Teaching..
The things I found most useful in my teaching had more to do with the structure of the discipline itself than with any particular piece of content. Two broad categories are:

A. **Seeing and exploiting structural similarities.** For example, the Chinese remainder theorem and Lagrange interpolation are really the same theorem (with the same proof), just stated in two different rings that share an underlying structure.
B. **Placing high school mathematics in the larger landscape of mathematics as a discipline.** An example is broadening questions about Pythagorean triples to include right triangles with *rational* (rather than just integral) side-lengths, a direction that leads to the frontiers of what's known in arithmetic.

In addition, there are profession-specific applications of classical and modern mathematics that are extremely useful for teachers. One of my favorites is the use of norms from number fields (or rational points on curves) to design problems for students that "come out nice" like integer sided triangles with a $60°$ angle or cubic polynomials with lattice point zeros, extrema, and inflection points.

Vignettes..
Bill's first one is one of my favorites, and a perfect example of the way high school connects to the broader landscape. Some ideas for others are already mentioned above:

A. The various ways to fit polynomials to tables, including Lagrange interpolation and Newton's difference formula.
B. An introduction to the theory of congruent numbers, generalizing Pythagorean triples to rational right triangles with integer area and the various equivalent formulations of the problem.
C. The use of algebraic number theory to design pleasing problems for high school students.

A.8 Dallas, Heather

I am a lecturer in the UCLA Mathematics Department. I teach courses for 1st year students and a hybrid capstone/pedagogy course for senior mathematics majors who plan to teach secondary school. I also serve as Executive Director of the Philip C. Curtis Jr. Center for Mathematics and Teaching in the UCLA Mathematics Department. For over a decade, I taught mainly functions, single variable calculus and series at the secondary school level. For several of those years, I also taught students struggling to pass a first course in algebra.

I have found the countability and uncountability of the rationals and the reals a nice means by which to illustrate the beauty and power of mathematics to non-mathematical friends. It was in my first year of teaching that I first spoke about this issue to a non-mathematician, a first year Algebra student who asked "Are there more decimals than fractions?

In the secondary school calculus class, a student once asked me if the Newton-Raphson method for approximating roots of functions could be used to find complex roots. This question led to an interesting multi-day investigation for the class, including work on related fractals (see http://mathworld.wolfram.com/NewtonsMethod.html). In the same class, another student asked me why his calculator produced an answer for 0.5!. Since the class was familiar with the integration techniques, we were able to investigate some aspects of the gamma function to answer his question. I recall these events to demonstrate that one area of interest to secondary mathematics teachers might be to extend to the complex number topics they teach for real numbers.

A.9 de Leon, Manuel

a) The non-Euclidean geometries and the understanding of the universe.

300 BC Euclid wrote The Elements, and stated the famous five postulates on which he based his geometry. The fifth theorem (as it was stated later by Playfair) says that: "Given a line and a point not on the line, it is possible to draw exactly one line through the given point parallel to the line." Many attempts were made to prove the fifth postulate from the other four, many of them being accepted as proofs for long periods of time until the mistake was found. D'Alembert, in 1767, called it the scandal of elementary geometry. Later, in 1823 Janos Bolyai discovered "a strange new world"; indeed, a new geometry was possible just assuming an alternative fifth postulate that gives origin to the non-Euclidean geometries (independently developed by Lobachevsky). The following work by Riemann, Beltrami and many others opened a new mathematics. This new geometries were used by Einstein to base his Relativity theory leading to a better understanding of our universe.

b) The quest for the Foundations of Mathematics.

At the beginning of the XX century David Hilbert addressed to the participants of the Paris ICM describing an enthusiastic panorama of the mathematics in these days. But the things were very different and during the coming years the mathematicians discussed the foundations of the discipline. This period of mathematics is not usually included in the syllabus of secondary teaching, and I think that it would be a wonderful motivation for the students. Fortunately, there is a fantastic recent graphic novel Logicomix inspired by this story. The narrator is the logician, philosopher and pacifist Bertrand Russell, and through the pages of Logicomix flow the developments and personal lives of Frege, Hilbert, Poincare, Wittgenstein and Goedel. This quest for the foundations of mathematics is the best issue to show how mathematics is more than a science, and a inalienable part of the human culture.

A.10 Doolittle, Edward

I am a Mohawk Indian from Six Nations in southern Ontario. I earned a PhD in Mathematics from the University of Toronto in 1997. My thesis, under the supervision of Peter Greiner, was on the subject of hypoelliptic partial differential operators. In addition, for several decades I have had an interest in mathematical problem solving, the popularization of mathematics, and in issues surrounding mathematics and Indigenous people.

Favourite piece of mathematics to tell friends about: I like to tell people about the mathematics of vibrating objects: strings, trumpets, drums, etc. One dimensional objects often have a nice combination of fundamentals and overtones that allow them to be played in harmony, but two dimensional objects don't: the dimension of the object determines properties of its spectrum, which for two dimensional objects is not suited to musical harmony. We can tell the dimension of an object from its spectrum, which raises the natural question of whether we can tell other properties from the spectrum, perhaps its exact shape.

There's so much mathematics I'd like secondary teachers to know. If I were forced to pick one, interesting applications of matrices come to mind, quantum computation and information in particular.

Suggestions for vignettes: 1) Cantor's diagonal method, the halting problem, use of similar methods in various applications, e.g., Conway's Game of Life, a problem similar to the Collatz conjecture. 2) The use of graph-theoretical algorithms to design word problems. 3) Proof of Euler's polyhedron formula using spherical trigonometry, and a similar analysis using the geometry of the torus. 4) The Cayley-Hamilton theorem, applications, and a one-line proof using complex analysis. 5) Slide rules; how to convert Celsius to Fahrenheit with a slide rule; conjugates; how to solve puzzles with groups and conjugates. 6) Maybe something to do with Hamiltonian mechanics of a bead sliding on a plane.

A.11 Epp, Susanna

1. As part of the lead-in to the workshop, we ask that you contribute some information that will help with the workshop preparations. Please write a paragraph or two about your mathematical experiences.

My most notable mathematical experience has been a steadily deepening understanding for the role of logic and language in mathematical reasoning. My father had been trained as a lawyer but then followed his heart's desire and became an English professor and biographer, ultimately winning several major awards for his writing. My mother studied Latin and Greek, did undergraduate and graduate work in English (including a full year of Old English!) and spent many years teaching English at a community college. Growing up in this environment, I absorbed attention to logical nuance and the precise use of language like mother's milk. So when I studied more theoretical mathematics, the language in which the abstract ideas were expressed, and the logic of proof and disproof came naturally, and the thought processes that were not already innate were acquired quickly and with enthusiasm.

As a relatively new teacher in the 1970s, I was surprised by how poorly many of my students performed in courses like abstract algebra and real analysis. At that time my department only required a year's worth of calculus plus one quarter of linear algebra – both heavily computational – as a prerequisite for advanced courses. So I worked on developing a course to help students prepare for them. Teaching this course in a highly interactive way was what started me on the road toward conscious awareness of the logic and language that underlie mathematical reasoning. Even now, I can be surprised by some of the locutions that students fail to interpret as a mathematician would expect, and only recently have I come to more fully understand the practical role played by some of the logical principles that logic textbooks state in starkly abstract ways.

My growing understanding of these issues has profoundly influenced both my teaching and the textbooks I have been writing, which try to address students' problems with logic and language as well as covering standard mathematical topics. I have not conducted formal research about the effectiveness of this approach, but I have been gratified to receive thanks for the way it is used from students and teachers around the world.

One of my current concerns is the many problems students have in working with literal symbols (or variables) in mathematics. I have published a paper about this issue (http://condor.depaul.edu/sepp/VariablesInMathEd.pdf) and another paper that is related to one aspect of it (http://condor.depaul.edu/sepp/ICMI_19_Epp_Issues_with_EI.pdf). I think the insights of 20th century logicians about variables are appropriate for consideration to be included in the Klein project.

Then also describe:

a) One piece of mathematics that is your favorite example to tell non-mathematical friends about the power, beauty, or applicability of mathematics.

It's too hard to choose just one piece of mathematics. I'd offer the proof that there are infinitely many prime numbers, two proofs that the square root of 2 is irrational (the one using the fact that if the square of an integer is even then the integer is even and the one based on the unique factorization property of integers), and the proof of the handshake lemma.

b) One piece of modern mathematics that you would like to know about (if you are a secondary math teacher), or that you would like people who teach senior secondary math classes to know about (if you are not one).

The proofs that the rational numbers are countable, that the irrational numbers are uncountable, and the related proof that given any computer language, there are not enough programs in the language to compute values for all possible functions from the positive integers to the set $\{0,1,...,9\}$.

Possible vignette: Tracing 'incompleteness' arguments: From the proof that there are infinitely many prime numbers (so any finite list is incomplete), to the proof that there are uncountably many real numbers (so the rational numbers are incomplete), to Goedel's proof that any consistent formal system rich enough to include arithmetic contains true statements that are not provable (so the set of provable statements is incomplete). Noting the roles of (1) constructing an object that shows the incompleteness of the system, and (2) the role of self-referential and diagonal arguments. To make this a vignette, just an rough outline of the Goedel proof could be given.

Possible vignette: Something about the evolution of the historical concept of variable.

A.12 Hsu, Eric

(b) One thread of mathematics I'd like to see secondary teachers know is the development of N from the Peano Axioms including inference of commutativity, associativity and distributivity, Z as the Grothendieck Group of N, Q as a field of fractions, and R as a completion of Q. An amazing return on a concise set of axioms and well-chosen definitions.

A.13 Iwen, Mark

I first became interested in mathematics when I took my first proof based class as an undergraduate. The course fulfilled a requirement of my computer science major, and I had no real interest in mathematics before I took it. In fact, when the course began I was planning on doing as little work as absolutely possible in order to get an A or B in the class. By the end of that semester I was a math major. The proof based course was my first experience with a mathematics class that did not revolve around tedious algebraic computations, and I thoroughly enjoyed it.

Before taking that required class my only exposure to mathematics had been the algebra, trigonometry, and calculus classes I had taken in my high school. Although I recognized these subjects as extremely valuable, I also found them extremely boring (at least as taught in my high school). It was only through shear force of will that I studied them at all.

In my personal opinion, there are other mathematical subjects which are not only at least as useful as calculus, but also significantly easier to teach well. Several of these general subjects include: graph theory and basic optimization, voting theory, introductory game theory, and linear algebra (as an alternative to calculus). All of these subjects have fairly transparent applications and are directly related to computing. Furthermore, the first three subjects can be taught, at least partially, by playing games! I would be very pleased if secondary teachers knew more about any of these four subjects.

One possible topic for a vignette could be sparse approximation in linear algebra (worked on recently under the moniker of compressed sensing). Related areas in mathematics and engineering include error correcting codes, probability, numerical linear algebra, and signal processing. The topic can be easily introduced through a set of problems known as “group testing” problems, which themselves ultimately stem from cost saving measures instituted in the screening of new army recruits for syphilis during WWII.

A.14 Landry, Therese

As a student of mathematics, I continually pursue opportunities to re-contextualize and redefine mathematical connections. While an undergraduate at Brown, I travelled to Budapest to engage in the Hungarian tradition of doing mathematics. Over the last two years, I travelled to Boston from Los Angeles several times to participate in the PROMYS summer program and its academic year workshops. I then went to Park City in like pursuit of learning mathematics. Leaving familiar spaces has always reoriented my perspective on the foundations of mathematics.

a) The following paragraphs describe a definitive mathematical encounter with a favorite example of the power and beauty of mathematics.

When I reoriented the focus of my undergraduate studies from particle physics to number theory, I resigned myself to the likelihood that these two interests diverged substantially. Mathematical physics speaks in differential equations, not prime numbers. Shortly before I graduated, a friend handed me an issue of the American Mathematical Monthly. Aware of my earlier background in physics, he directed my attention to an article on a correspondence between zeta functions and quantum physics. At long last, my parallel loves had met.

Properties of primes define number theory. Indivisible and elemental, they uniquely compose each integer. Zeta functions powerfully sum up celebrated results about prime numbers. Though defined as an infinite series involving each integer raised to the same power

$$\zeta(s) = 1 + 1/2^s + 1/3^s + \cdots + 1/n^s + \dots,$$

a zeta function can be rewritten as a product involving each prime number

$$\zeta(s) = (1/(1 - 1/2^s))(1/(1 - 1/3^s))(1/(1 - 1/5^s)) \cdots (1/(1 - 1/p^s)) \dots .$$

The unboundedness of this function at $s = 1$ thus proves the infinitude of primes. At $s = 2$, the function curiously evaluates to $\pi^2/6$ and describes the probability that any two integers are coprime, or indivisible by any of the same primes. Extending the values of s from the real line to the complex plane opens up another dimension in the geometry of numbers. Primes occur at random throughout the real numbers. Sifted through Riemann's extension of the zeta function, they seem to fall on a straight line running through the complex numbers. Riemann claimed that all primes map to this critical line. Many results in modern theory depend on the truth of Riemann's Hypothesis.

Famously over tea at the Institute for Advanced Study, Dyson and Montgomery uncovered a connection between the Riemann Hypothesis and the energy levels of heavy atomic nuclei. The energy levels to which an atom can be excited uniquely characterize each element. The Schrodinger equation forms the basis of quantum physics and a means of calculating this spectrum of values. Computationally and experimentally, determining the spectrum of a complex quantum system, like the sixty-eight-proton-nucleus of erbium, is notoriously difficult and is an active area of research. Dyson noticed that the distribution of energy levels predicted for such heavy atomic nuclei followed the pattern Montgomery predicted for the distribution of primes along Riemann's critical line. Here was symmetry bared. In mathematics, truth and beauty, so seemingly parallel, meet.

b) I would like to know if any interdisciplinary work is being done in number theory and evolutionary biology. I would also like to learn about current research problems in the geometry of numbers.

A.15 McCallum, William

I am a mathematician with an interest in mathematics education. An early mathematical experience was discussing with my mother why the sum of two odd numbers is even. I used to enjoy logic puzzles, but was never particularly interested in school mathematics. My eyes were opened to the beauty of mathematics in my first year in university, where I learned calculus out of Spivak's book and read Lang's Algebra. I was inspired by my late mentor, Alf van der Poorten, to study number theory. I remember a couple of wonderful moments of discovery as an undergraduate: one that comes to mind is figuring out that the only automorphism of the reals is the identity but that the complex numbers have very many automorphism. I came to the United States to pursue a Ph.D. in mathematics, and have for a long time been interested in the interplay between algebra and geometry. For the last 20 years I have also had a strong interest in mathematics education, at both the university and K-12 level.

There are many pieces of mathematics that I have told my non-mathematical friends about over the years, but most recently I I like to explain the mathematical construction of the large chiral polyhedron in my front yard. It is constructed by taking an equilateral triangular lattice, taking the sublattice of covolume 7, folding it up into an icosahedron, the pushing the vertices of the original lattice out to the enclosing sphere and erasing the edges of the icosahedron. Easier to see than to describe: visit http://philomorph.net/dream/.

As for a piece of modern mathematics I would like secondary teachers to know: I can think of lots of examples that are not modern (Galois theory, some simple algebraic number theory, symmetry groups of geometric objects ... many of the things in Klein's book). The pieces of modern mathematics that teachers should know tend more towards the applications of mathematics, some of which are already captured in the vignettes we have posted (e.g., Google's page rank algorithm). Since the modern world is so drenched in computation, I think it would be useful for teachers to know something about algorithms; for example, an appreciation of why algorithms for multiplying numbers are so much faster than the algorithms for factoring them, and how this leads to the trapdoor encryption schemes on which modern commerce is built.

A.16 Metzler, David

I have a somewhat unusual background for a secondary mathematics teacher. I received my BA in math in 1992 from Rice University and my Ph.D. in math in 1997 from M.I.T. I then did three years as a postdoc at Rice and four years as an assistant professor at the University of Florida, working in symplectic geometry, before switching to secondary school teaching in 2004. Since then I have taught at my alma mater, Albuquerque Academy, in Albuquerque, New Mexico.

I have really enjoyed focusing on teaching in the high school environment. I appreciate the closeness I have with my students (especially with relatively small classes of 10-20), something I did not have at the college level. I teach primarily precalculus through multivariable calculus, with excursions downward to Algebra II and up to linear algebra, differential equations, intro analysis, and various other courses done as independent study or reading courses.

It's exceedingly hard to pick just one piece of mathematics that I like to tell people about. I regularly give lectures to community groups on various mathematical topics, and I'm currently working through a series of talks on the Clay Millennium Problems. (See for example http://www.youtube.com/watch?v=a8AO_Q8ANl4 about the Riemann Hypothesis, although I'm afraid Prof. Conrey might take issue with the many liberties I take with that particular subject!) However one thing I particularly love telling students and community groups about is the formulation of special relativity as Minkowski geometry – it's such a beautiful, unexpected, and deep connection, with continuing relevance for physics.

As to one piece of mathematics that I'd like to know more about, that's difficult to choose as well. The subject of nonlinear PDEs comes to mind, since I have a former student who is now trying to apply them to study cell membrane behavior related to Alzheimer's Disease. Also, I'm going to need to understand them well enough to give a general audience talk on Navier-Stokes next month! In short, it's a subject that's outside of my expertise, that has deep pure mathematical meat to it, and is very applicable.

A.17 Mumford, David

a) You asked for a paragraph or two about your mathematical experiences. My career has been split between pure and applied math, both of which I love deeply. At some point in the mid 20th century, the two split and became separate disciplines in the sense that practitioners of each rarely read results achieved in the other nor do they share the same basic knowledge. I hope the Klein project balances the two.

b) Here's an example to tell non-mathematical friends about the power, beauty, or applicability of mathematics. I have used this in a class for non-math majors:

"Your second assignment is to measure the size of the earth using two photos of the Newport bridge, one taken from 18.5 miles down Narragansett Bay (near Providence) and the other from close up:

[your website won't accept an image]

You need to know: (a) the tower of the bridge is about 405 feet high and (b) the camera was approximately 25 feet above the water when the first photo was taken. Also note that in this photo, the black is open water while the distant water is frozen (and there is a mirage).

So estimate the true horizon by extrapolating the faint line jutting out from the land on the right. You may want to print out these pictures and use a ruler to measure parts of the tower in each shot. The diagram here should help you put all this together.

[2nd missing image]"

The point is that one can use Pythagoras's theorem (without any trig) to solve for the size of the earth when one knows how much of an object, at a known distance away, is cut off by the horizon. It gets a bit trickier when you allow for how far above the sea you are standing – now a bit of algebra is needed too.

c) You asked for One piece of modern mathematics that you would like people who teach senior secondary math classes to know about. I suggest the wonderful paper of Claude Shannon "Prediction and Entropy of Printed English". This is basically elementary (though his paper is a bit formal). He devises a wonderful parlor game whereby you can estimate how many yes/no questions are needed per letter to communicate an English text. (The answer: about 1.) The whole thing introduces you to the fact that information can be measured numerically, just like distance and time. The paper is online at Bialek's website: http://www.princeton.edu/∼wbialek/rome/refs/shannon_51.pdf.

A.18 Recio, Tomas

1- Please write a paragraph or two about your mathematical experiences.

Professor Tomas Recio (Oviedo, Spain, 1949) holds a Ph.D. in mathematics from the University of Madrid (1976) and the Chair of Algebra at the University of Cantabria since 1982. Along the years, he has been involved in different academic responsibilities as Secretario General, Vicerrector de Investigacion, Director of the Institute for Educational Sciences (ICE), etc. He has held different visiting or short-term positions at a number of universities and scientific institutions in Spain, Europe and USA.

His research topic is algorithmic real algebraic geometry, with circa 200 published contributions and 15 Ph.D. dissertations as advisor. He is (and has been) member of different editorial boards and committees for international conferences on this topic. Has been involved, often as main investigator, in circa 20 research projects, at Spanish or European level, on pure or applied mathematics, including mathematical education. He has been involved, as well, at different levels of responsibility, in the scientific assessment of research projects, academic curricula and positions, in many different countries.

Professor Recio has been the president of the Comision de Educacion de la Real Sociedad Matematica Espanola (1999-2006), president of the Spanish Sub-commission of the International Commission on Mathematical Instruction (ICMI) (2002-2006), and president of the Consejo Escolar de Cantabria (1999-2008). Has published a number of papers on mathematical education and its situation in Spain.

More information at www.recio.tk

2-Describe:

a) One piece of mathematics that is your favorite example to tell non-mathematical friends about the power, beauty, or applicability of mathematics.

Automatic discovery of elementary geometry theorems http://dl.acm.org/citation.cfm?id=1574589

b) One piece of modern mathematics that you would like to know about (if you are a secondary math teacher), or that you would like people who teach senior secondary math classes to know about (if you are not one).

Computational Geometry http://en.wikipedia.org/wiki/Computational_geometry

3- Read the introduction at http://kleinproject.org, and then click on the Vignettes link and read one of the first three vignettes listed there.

Please suggest a few possible topics for vignettes

For instance, those mentioned above.

Also, please share any other thoughts you might have about the Klein Project, its aims, and its products.

Just one thought: the Klein Project is not (only) about dissemination of XXth Century Mathematics.

A.19 Rosenberg, Gabriel

My father was an algebraist, and I first strived to learn more about mathematics in an effort to understand his work. As an undergraduate I took as many math courses as I could in addition to participating in summer programs such as PCMI. I initially went to graduate school with a thought to studying knot theory, but soon discovered the wonders of Bass-Serre Theory. I completed my dissertation, Towers and Covolumes of Tree Lattices, under the guidance of Hyman Bass in 2001 and from 2000-2004 collaborated regularly with Lisa Carbone in the study of tree lattices.

In 2004 I shifted tracks and went to teach full time at Bard High School Early College, a public high school in New York City where students earn both a high school diploma and an Associates Degree in four years. Working with other teachers there we designed a unique curriculum where students in 9th and 10th grade could learn the essentials of algebra, geometry, and functions that they would should they wish to take a course in calculus. Many of our students do take calculus and sometimes linear algebra and ODEs as well in their last two years at our school. I enjoy, though, looking to other college and universities for college courses that can appeal to students who do not want to take that route. We now have an introductory course on the concept of groups using a book by David Farmer, and one introducing the concept of topology based on a book by Farmer and Stanford. Ive also introduced a course looking at the applications of math to cryptology, and another looking at applications to political science.

When non-mathematical friends ask about the beauty of mathematics, Im inclined to talk about complex numbers. Heres a concept that seems far removed from our actual experience at first (we even contrast them with the real number system), but upon further exploration turn out to be quite natural and incredibly useful. I especially love using them to cement the relation between algebra and geometry. Translation is accomplished by addition, dilation by multiplication by positive numbers, reflection by multiplication by -1. It seems like there should be some algebraic tool to accomplish rotation, and with complex numbers there is!

Im not quite sure how to answer the last question as I consider myself both a secondary math teacher, and a post-secondary math teacher. I suppose Ill answer it both ways. Id like to know more about Lie groups. I certainly know something about them, but I dont have nearly the sort of understanding that I would like to have. I would like more secondary school teachers to know more about number theory. Its an area filled with so many challenging (and even unanswered) questions that a high school student can understand and find interesting. Furthermore, the attempt to answer some of those questions can motivate the need for definitions and proof, as well as highlight how seemingly remote areas of mathematics can be intertwined.

A.20 Rouleau, Etienne

I am a teacher in mathematics in College Jean-Eudes, a private school of Montreal. I teach in secondary third and fourth with from 14 to 16 years old students. The main concepts in the study are relative to the functions, to the algebra calculation (operations on polynomials and factorization) and in the geometry (area and volume of solids, isometric triangles and similar triangles, analytical geometry). Also, I give courses at the University of Montreal. I give courses of didactics to future teachers. I am interested in the Klein Project because I like being able of showing to my students the links existing between what they make in class and the current utilities of the mathematics. I like managing my students towards problems of enrichment and motivating them to go farther in their reflections. I hope that I can make an interesting contribution to the Klein project.

A.21 Rousseau, Christiane

My background and interest in the workshop.

I am a university professor. My research specialty is dynamical systems. But, I have specialized my teaching to future high school teachers. In my courses I like to make them feel that mathematics is a living discipline within science. I want to make them feel how the mathematician works, and also how a breakthrough can come from a simple idea. I am quite involved in popularization of mathematics. I regularly give lectures in colleges (cegep level) or during congresses of teachers. I organize the public lectures series at CRM. I am also an active collaboration of the magazine Accromath (www.accromath.ca), which is distributed freely in schools in Quebec. I am a member of the design team of the Klein project.

I have two favourite examples to tell non-mathematical friends about the power, beauty, or applicability of mathematics.

The first one is the Banach fixed point theorem. It is extremely easy to illustrate what it means if we play with a lid of "The Laughing Cow". At the same time, it is one of the most powerful theorem of mathematics with many applications in diverse fields such as differential equations and imagine compression through iterated function systems. This last application has been illustrated in my vignette "Banach fixed point theorem and applications".

My second favourite example is Kepler's conjecture. This conjecture states that the densest packing of spheres is that which you observe with oranges at the fruit market, and which is called face-centered cubic lattice. This conjecture, that everyone can understand without any mathematical knowledge was proved only very recently. Each time, we take a regular packing of spheres it is possible to prove that it is less dense that the face-centered cubic lattice. But, what about irregular packings? Can we do better? The challenge is to explain what could be the idea of a rigorous proof covering the non regular cases. This can be done through the two-dimensional problem of the densest packing of disks in the plane. It is also fascinating to describe applications of this problem, and the related application driven research problems. I have written a vignette on the topic: "Kepler's conjecture on the packing of spheres"

Here are potential themes of vignettes that could be associated to the analysis chapter of the projected book:

Potential vignettes related to the analysis chapter:

- A vignette on Brouwer's fixed point theorem and its application to economy: existence of Nash equilibria. Another vignette with the link with Spencer's lemma. This yields a method for finding the fixed point.

- A vignette on Morse theory

- A vignette on the mathematics of the CT-scan : again a decomposition of the information in an infinite basis

- Equivalence between the Brouwer's fixed point theorem and the theorem of Borsuk claiming the non existence of a retraction from B^n to S^n. Borsuk's thm : it is impossible to brink the skin of a drum to the edge without tearing it. Applications of Borsuk's thm

- Borsuk-Ulam thm : if $F : S^n \to R^n$, then there exists p such that $F(p) = F(-p)$. Application : cutting two pancakes with one knife cut.

- Can we hear the shape of a drum ? A long standing problem whose answer was finally proved to be negative

- Enrichment on wavelets

- The Poincare index of a singular point of a planar vector field. Link with the Euler characteristic : the sum of the Poincare indices is equal to the Euler characteristic.

- Definition of dimension. Fractal dimension. Some applications.

Some classics that could be the subject of a vignette:

- RSA cryptogrphy

- The idea of a probabilistic algorithm for primality

- Zero-knowledge proofs

A.22 Russell, Hilton

Mathematical experiences:

My education in mathematics at M.I.T. was perhaps a standard one, though my application of it afterwards was not. My high school did not offer Calculus, so I began my college career with Calculus and Differential Equations. My first courses beyond Calculus were one in abstract algebra and one in complex variables. My favorite courses were in analysis and an algebra seminar on non-commutative rings. I became a Calculus tutor for the math department in my sophomore year, and taught recitation sections of Calculus in both semesters of my senior year. I entered the U.S. Army upon graduation in 1983, serving as an Air Defense Artillery officer for seven years before being sent to obtain a Masters Degree (Applied Math, RPI, 1992) in order to teach Calculus and Differential Equations at West Point. In grad school, I enjoyed studying computational complexity and integer programming. While at West Point, I met John Dossey, Frank Giordano, Chris Arney, and Don Small, and due in part to their enthusiasm for math education, decided to leave the Army and become a teacher. My high school teaching career has been exciting for me. My experiences in graduate school with mathematical modeling (Don Drew) and at West Point with applications projects have strongly colored my teaching. In every course I have taught, I have developed projects requiring students to synthesize and apply the mathematics they have learned. I enjoy studying pedagogy and assessment, especially regarding gifted and talented students and applications of technology in teaching. For several years I have found summer opportunities to work with other teachers and both do some math and think about teachingthrough the Math Forum, NCTM, or, this summer, the Park City Math Institute.

One of my favorite math ideas to share is the ideas of examining cross-sections of common objects. People think that slicing a cube will always produce a square, or a rectangle, or maybe a parallelogram; some can see a triangle. Few think about the possibility of a pentagon or a regular hexagon. If I ask them how to cut a cube to reveal the regular hexagon, people pause, think, and occasionally have a glimmer of insight and experience the joy of abstract visualization. When they have there aha! I point out that mathematicians look at all sorts of questions hoping to have that same experience.

I previewed two of the sample vignettes, Heron Triangles and Elliptic Curves and Higher Dimensions. I found both to be accessible and interesting (though typesetting/browser problems and some sign errors in the first were distracting).

Prior to reading these vignettes, I was thinking of elliptic curves as an interesting source for exploration. They do provide some entry into the proof of Fermats last theorem, as well as many other connections (the idea of the secant method for generating rational solutions, for example).

The use of quaternions and octonions, as well as other algebraic geometry applications, might be interesting; I have explored these a little, but dont have much experience with the topics. The use of linear algebra and coding theory in 3-D animations, special effects, and light staging (work of Paul Debevec, for example) is another very interesting topic.

Im very excited about the goals of the Klein project. I majored in pure math, but I discovered I love working with young people more than doing research, so I became a secondary teacher. I still appreciate the beauty of the structures underlying mathematics, and love being able to share that appreciation with my students. Reading Ian Stewart, or about Nicolas Bourbaki, helps to keep my enthusiasm alive. To give secondary teachers access to vignettes about current, leading-edge math research, written to be both accessible and mind-stretching, is a great opportunity to enrich math education.

A.23 Sangwin, Christopher

** My mathematical experiences.

From an early age I've always been interested in mathematics. For years 5 6 I was lucky to be in a very small village school of about 25 students. I think I was the only person in my year group, at least I do remember spending a lot of time working on my own, on a "BBC B" microcomputer playing with logo and writing programmes of my own.

At secondary school I was very lucky in years 12 13 to be in a very strong peer group and to have two very committed and talented teachers. They were "chalk and cheese" but the combination suited us well. One aphorism I learned has stayed with me: "The twin keys to high-school mathematics are, 'factorise or die' and 'the difference of two squares'". Not bad advice really.

I progressed to Oxford, then to a PhD at the University of Bath before taking up a position at the University of Birmingham. I've been lucky to collaborate with John Bryant in recent years from whom I've learned a great deal, about mathematics, engineering and a host of other things.

** A favorite example to tell non-mathematical friends.

Just the one example?! Ok, I'd tell non-mathematical friends about "shapes of constant width".

If you have a circle, then the width is constant. If the width is constant, then what can you say about the shape? The surprise, and most favorites need to have an element of surprise, is that non-circular shapes of constant width exist.

What is the width? It is the difference between parallel tangents, or parallel support lines to a convex planar figure.

The simples is Releaux's rotor. Start with an equilateral triangle. Draw the short circular arc, centred at one vertex passing through the other two. Repeat this to get three circular arcs. This has constant width as a tangent to one of the arcs is a constant distance from the opposite vertex. Easier to see than describe, and best to MAKE.

This can also be done in 3D.

** Modern mathematics I would like people who teach senior secondary math classes to know about (if you are not one).

Polynomial Grobner basis techniques. This links Gaussian Elimination and the Euclidean Algorithm. They enable all sorts of problems to be “solved”. Really amazing. It also explains why polynomial long division is so important....

Advanced, yes, but we didn't say high school students had to learn it!

A.24 Seidell, Debbie

My name is Debbie Seidell. I work as a high school math teacher, currently at the Horace Mann School in New York. I have eight years experience teaching in four different independent schools in New York and Massachusetts. I have taught every class in the standard sequence from sixth grade arithmetic through multivariable calculus. I also spent three years in the PhD program at the University of Washington in Seattle. I mostly likely would have focused on algebraic topology or Lie groups if I had continued with my studies. Unfortunately, math knowledge leaves you quickly once you stop using it all the time, and I have forgotten most of what I once knew. What I still carry from those three years is a good approach to problem solving, and the ability to recognize solid reasoning when I see it.

When I try to explain the beauty of mathematics to other people, I often start by telling them about one common scene from my grad school days. A small group of students work away at a problem for several days, culminating in a several-hour long work session and, at last, a viable proof. Exhausted, they lean back in their seats and say, “We got it. It's done, and it's correct. But it's hacked together, awkward and ugly. Now let's find a better way to prove it.” Being a mathematician felt like being an artist, always searching for beauty.

When I try to give an example of a beautiful piece of math, I usually start by talking about infinity, and how infinity can swallow addition and multiplication by finite numbers. The standard Hotel Hilbert and Cantor's Diagonalization Argument stories usually work very well. If I have more time, and a piece of paper to write on, I will talk about roots of unity. I talk about multiplication in terms of rotations and dilations, and lead into the beautiful symmetry of the roots of unity. Sometimes, I will talk about the non-orientability of the Mobius strip, or how adding a point to the plane produces a sphere.

As for what I would like to learn about, I would like to know more about fractals. I have learned some basics a few times over, but somehow it never stays with me. The idea of self-repeating patterns is great, but what does that have to do with the process that produces the Julia Set?

A.25 Weigand, Hans-Georg

I studied mathematics and physics. In mathematics my main interestes during my study have been complex analysis, especially Riemann surfaces and differential equations in the complex plane. I have done my dissertation in mathematics education concerning the learning of conepts and the influence of new technologies on this learning process. I am professor for mathematics education at the university and my main subjects are concept formation, the teaching and learning of geometry and the influence of ICT on teaching and learning in mathematics classrooms and in teacher educaiton.

Favorite example:

Mathematics of the honey bees. How do the bees create their waves and how do the wave look like they look.

Would like to know about:

Security codes, QR-codes.

A.26 Wiegers, Brandy Sue

I am currently working at San Francisco State University as a Project Director of Outreach and Student Success at the Center for Science and Mathematical Education (CSME). Through this position I am blessed with weekly, if not daily, mathematical experiences that challenge my understanding mathematics as I try to share the topic with others. Students can look at mathematics with such fresh and new ways and that is inspiring.

I came to SF State through a series of opportunities where I left myself open to exploration of as much math and science as possible. I have a Bachelors of Biological Systems Engineering and a BS in Mathematics, my PhD is in Applied Mathematics, and I've done work in prostate cancer research and on the symbiotic relationship of algae and sea anemones. Most recently I'm involved through SF State and MSRI in Math Circles, a math GK12, and many other informal math education opportunities for students and teachers.

In my career I've had the chance to be more and more involved in sharing beautiful and inspiring mathematics with students and teachers. From this it's become a job reality that I'm constantly finding math to be shared with friends and colleagues. One of the most recent examples has been looking at the great mathematics used to make computer graphics more realistic. For example the use of the Golden Ratio in creating the alien Avatar plants that are alien but still recognizable as plants is amazing. In addition to looking at this mathematics I've also been looking back to the mathematical lore and tradition of our community. Slide Rules have long been associated as mathematical tools but until this year I had never touched one. I've been sharing these tools with students and teachers and loving the experience of re-discovery of this tool.