Some considerations about the advances of Mathematics in 20th century, and the possible implications to modernize the school curriculum.[1]

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ABSTRACT: This note intends to summarize the ideas presented in the talk during the 1st Workshop of Klein Project - Funchal, Madeira ([1]). It aims to comment on some aspects of the advances of Mathematics during last century that could modernize the school curriculum, based on the reflections of the author. The considerations are not exhaustive, but they focus mainly on the topics of Geometry, especially those conveying the concepts of manifolds and coordinates that could be translated into school curriculum. Also, we argue the role of Linear Algebra as fundamental language of modern mathematics and as a basic tool in applications such as computer graphics and robotics.

INTRODUCTION: When Felix Klein published his books "Elementary Mathematics from Advanced Standpoint", his intention was to provide the secondary school teachers with the fundamental ideas of mathematics, showing "the mutual connection between problems in the various fields" taking the teachers to the knowledge of the advances in mathematics that they had missed because of the discontinuity between the instruction and the research development. As Barton says in [2], the aim of the Klein Project is "to relate a comprehensive view of the field of mathematics to the content and approaches of senior secondary and undergraduate mathematical curricula". Based on this approach to Klein Project, the main objective of this note is to filter important concepts in the field of Geometry that have contributed to the advances of research in the 20th century and should be integrated into school curriculum in a meaningful way.

The table of contents of the Volume 2 of Klein's book shows already the idea of *manifolds*, a fundamental concept of modern mathematics of late 19th century and that gained definite importance in 20th century together with the consolidation of topology. Actually, if we look at the classical division of areas in Mathematics, such as Analysis, Geometry and Algebra, we notice that at the same time the areas of research in Mathematics have multiplied and specialized to yield new areas such as Computer Science and Statistics, more the areas have been intertwined to produce new findings and knowledge.

The following table is a sample of such branches of research fields in Mathematics that have developed in 20th century, and some of them have evolved from more basic fields to become independent and productive research areas themselves. One of most representative of this kind is Computer Science, along with Cryptography and Statistics. In pure mathematics, we see the development of important researches in Dynamical Systems, Differential and Algebraic Topology, Differential and Algebraic Geometry, to mention a few more related to the area of Geometry. In the table, we can see the fragile lines that

[1] This note elaborates some of the ideas presented by the author during the First International Workshop of the Klein Project, in Funchal -Madeira, Portugal, October 2009. Since then, The Klein Project in Portuguese on diverse areas of mathematical knowledge has been developed in Brazil through workshops with researchers and teachers. (Comments in September 2011).

divide several research topics, for they intertwine each other to produce more knowledge as well as technology.

The reflections on school curricula are slower than the discoveries and applications, but nevertheless they happen, not always with clear agreement about what and how the scientific advances should affect the common knowledge.

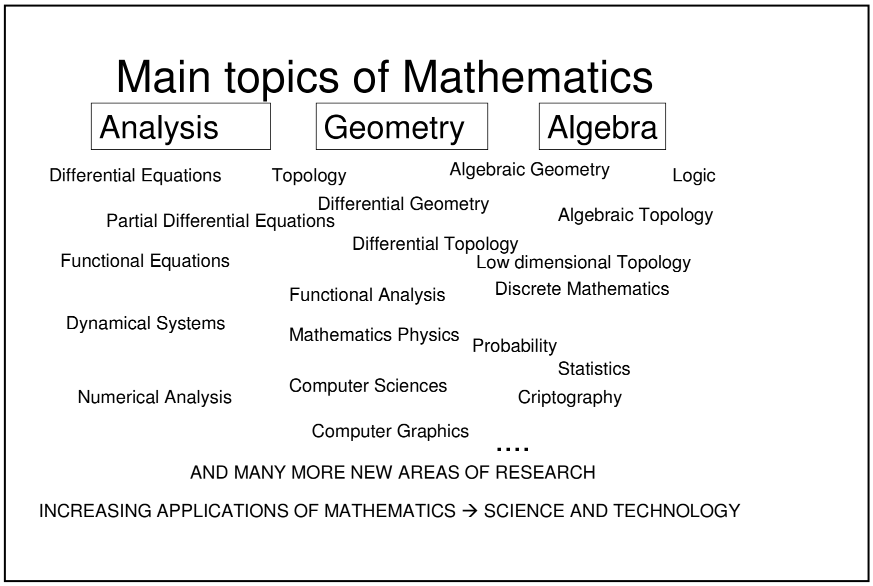

Table 1: Some branches of research fields in mathematics of 20^{th} century.

Looking at the Table 1, we could ask what of these topics is familiar to our school teachers with degree from teacher preparation courses. Of course, they do not need to know them as researchers, but they do deserve knowing about the missing bridges that separate the school topics and the world of mathematicians.

Although the intention of Klein was to present the modern ideas to school teachers through the links in the problems, we see that the school curricula are far from making connections among the fundamental areas of mathematics. Many topics taught at secondary schools are not understood by most students and some teachers about the origin and the evolution of the concepts, their importance for the discipline mathematics and the connection to the real context in their careers or at the society. As an example, we can cite the difficulty of students to understand the concept of "dimension" that connects the geometric and algebraic approaches. Other significant example is the difficulty to understand the concepts of "parameters and variables" when dealing with functions, algebraic expressions and graphics.

In this note, we first comment about some aspects of the research in Geometry to justify our proposal of the topics and the approaches to modernize the school curriculum.

MANIFOLDS: The concept of a manifold (topological or differentiable) has been developed as fundamental to modelize the various objects of study throughout the 20^{th} century. We will not explore here the rigorous mathematical definition of a manifold, because it will deviate from the objective of this note, but we will rather comment on the basic aspects that remain true to be transmitted to secondary school level students and teachers to perceive their importance in the curriculum.

Basically, the idea of manifold generalizes the visual geometric objects that secondary students can understand, namely curves and surfaces. For centuries, the Geometry was studied under euclidean viewpoint, and since the introduction of *coordinate systems* to describe geometric objects (Analytic Geometry) more refined techniques have yielded new and strong discoveries that changed the history of mankind. Although a study about the origins of the concept of n-dimensional objects goes back to 18^{th} and 19^{th} centuries, Riemann (1856) was the first to do a generalization of a surface to higher dimensions and introduced the term "manifold". Riemann's first ideas evolved into the modern formalization of a manifold. It was during the 20^{th} century that the definition of manifold and clarification of its foundational properties have been firmly established and understood, so that the concept of a manifold underlines, today, most objects of research in pure mathematics.

The establishment of the concept of a manifold and of the transformation groups has been essential to the advances of Mathematics and its applications, especially for physics (e.g. the models of special relativity and of the general theory of relativity) and other new branches of mathematics.

The definition of manifold brings meanings to a generalized concept of *coordinates and parameterization mappings*, freeing the classical differential geometry of curves and surfaces of 3-dimensional Euclidean space implying new ideas about *dimension*.

Manifolds also play important role in the transition from early models of non-Euclidean geometries to advances on synthetic model of an object, with topology induced by atlas of local coordinate mappings with open regions of R^n (or C^n), and concepts of vector fields, without considering an immersion in ambient space. In this aspect, Linear Algebra is basic and important tool to start developing results on local geometry of manifolds.

The evolution of calculus in n variables to analysis on manifolds permitted the generalization of the concept of derivation and subjacent structure of tangent space, which algebraic dimension relates to the geometric dimension of smooth manifolds. Then, the evolution of the theory of integration of calculus to the integration of differential forms on manifolds conducted to the generalized form of fundamental theorems.

The development of research on the geometry of manifolds has shown strong progress, and the interconnection between the main areas of mathematics is seen in Geometry through Topology for global results and Analysis for local properties.

The insight of Felix Klein of regarding the geometry from the perspective of transformation groups, and group actions with the invariance under an action as the characteristic property of one model of Geometry has evolved into the concept of principal bundle, Lie groups and Lie algebras, with many applications.

We do not go further in tracking the presence of the concept of manifold in the advances of mathematics as discipline, because the following questions come up to the attention:

- What is important to modernize the school curriculum of mathematics that filters such advanced knowledge?
- How could these concepts be worked out to become the basic knowledge needed to educate citizens of new times?

SCHOOL MATHEMATICS: In general, the mathematics curriculum of secondary schools encompasses algebra, trigonometry, functions, combinatorics and matrices with mechanical memorization of procedures and formulae. The topics of geometry often work the calculation of areas and volumes, missing the opportunity to connect them to the abstract reasoning of geometric properties that could help clarifying the role of modelization and deductive reasoning as fundamental abilities sought by School Education. Frequently neither the students nor teachers do know the importance of the topics they study to understand the world they live in. Another important observation is that the discrete mathematics, so present and important in modern applications, is a topic familiar at the elementary school (indeed, the counting positive integers is the predominant mathematics activity at this level). Yet, the gap between this level and the secondary level instruction when the topic of combinatorics is taught is very large, probably because the instruction is not based on "counting principles with understanding" but on the memorization of complicated formulae".

Regarding this gap, we strongly suggest the introduction of graph theory from early years of elementary school, starting with tree-representation of multiplicative principle of combinatorics, for this facilitates the work on discrete mathematics and applications.

Considering the advances of geometry as important research field in Mathematics that has much contributed to the recent achievement of Technology, it is very convenient that the secondary teachers and students know the modern approach to the concepts of "dimension" of objects (real and mathematical) and of "coordinates and parameters", since these are the basics for understanding the modern applications in Sciences and Technology.

The approach we are talking is the one that is underlined in the definition of manifold, which took many years to be as we know today, yet it is based on the first ideas of generalization from the intuitive ideas of elementary geometry. We could always start with examples taken from familiar objects from plane and spatial Euclidean geometry.

The possible implications of modern mathematics to school curriculum are proposed below, first inside the curriculum of preparation courses for future teachers of secondary schools:

1) Study the local parameterization of plane/spatial curves and surfaces as coordinate mappings to understand the idea of manifold and of dimension of geometric objects.

Example: The image of the application $\alpha(t) = (3*\cos(t), \sin(t))$, $t \in (\frac{\pi}{4}, \frac{7\pi}{8})$ is part of an ellipse, and it is described injectively by the *parameter* t, as it varies in the given interval. Check the Figure 1.

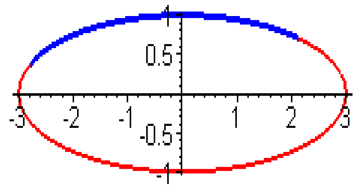

Figure 1: Part of Ellipse parameterized by 1 independent variable.

The whole ellipse is an example of 1-dimensional manifold immersed in R^2. Given any point on the ellipse, there is an open disc centered in this point that intersects the ellipse as the image of an injective parameterization defined like above that depends only on 1 independent variable, **t**. The collection of coordinate mappings (local parameterizations) whose images are obtained in this way and cover the whole ellipse is called an atlas, and some convenient properties on the intersections of these images entitle the concept of immersed manifold in the plane. In the same way, a helix on a cylinder is a good example to understand the concept of dimension related to the fact that, intuitively, the curve comes from a 1-1 deformation of a piece of line segment that can be described by only 1 independent variable.

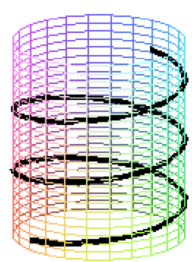

Figure 2: A circular helix on a cylinder.

Analogously, we could work out some examples of surfaces as 2-dimensional manifolds immersed in R^3, by connecting the concept of dimension of geometric object with the number of parameters of local coordinate mappings, e.g. the geographic coordinates of Earth on the sphere model. The smoothness of the parameterizations and of the change of variables on the intersections of coordinate neighborhoods can be associated to the smoothness of the geometric objects and the existence of tangent space (line or plane, accordingly) at each point. The relation between the concept of geometric dimension and of the algebraic dimension of the tangent vector space can be worked out in this way.

Key Example: Consider a parameterization g(t) = (2*cos(t) + t, sin(t)) defined for all $t \in R$. Part of the image g(R) is like in the Figure 3:

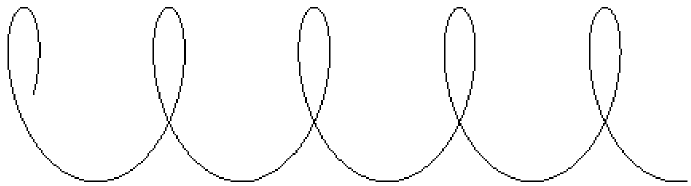

Figure 3: g(R) is not a manifold.

Although the parameterization depends only on 1 independent variable, g(R), as subset of points of R^2, is not a 1-dimensional immersed manifold because at the points of intersections, as one can see, no open disc centered in these points can produce coordinate neighborhood on g(R) described by 1-1 parameterization defined on a single connected interval. Other interesting example to introduce the concept of topology induced by the ambient space is given by the function $f: (0, 0.5) \subseteq R \to R$, $f(t) = \sin(1/t)$, that gives a local 1-dimensional parameterization to its graph, $G(f) = \{(t, \sin(1/t)): t \in I\} \subseteq R^2$. But the whole graph IS NOT an immersed 1-dimensional manifold of R^2, since open discs centered in points sufficiently close to the closed interval [-1,1] on y-axis cannot produce neighboring points on the graph described by 1-1 parameterizations defined on a single connected interval.

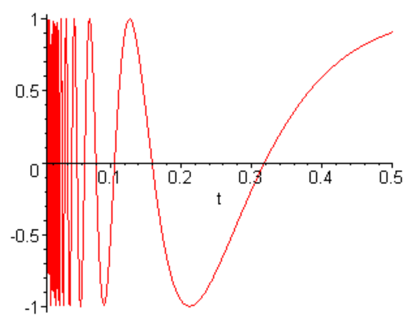

Figure 4: Graph of f(t) = sin (1/t).

This phenomenon is determinant to introduce the role of topology in the study of geometry, in teacher preparation courses. The use of technology is a powerful asset to present this example to secondary/undergraduate students in a modern way. The first ideas of topology by means of relative topology of immersed manifolds in ambient space diminish the gaps between intuitive ideas and those advanced and abstract generalizations of research mathematics.

2) Use the Linear Algebraic approach to understand the idea of dimension of vector spaces.

When learning the extension of ideas of dimension of a curve or a surface, as smooth manifold by means of the number of independent variables in coordinate mappings, we have already mentioned the idea of relating this to the existence of a tangent vector space (tangent line or plane, respectively) at each point, thus connecting the primary idea of dimension of a line or a plane to that coming from coordinate mappings. In this way, the idea of coordinate systems to understand the dimension of a manifold entails the role of matrix algebra as a tool to understand the geometry of the change of variables. The generalizations of derivation and integration on curves/surfaces, and then manifolds, are based on the first understanding of local linear structure, and the language of modern geometry starts with linear algebra and topology.

The introduction of geometric Linear Algebra as basic tool of modern mathematics from early years of undergraduate education can also enrich the secondary school classrooms as the teachers at this level could adopt as teaching methodology the geometric interpretation of algebraic operations of matrices.

We suggest that the approaches in Linear Algebra courses for teacher preparation programs should contain:

- ✓ The geometric interpretation of algebraic dimension of solution-spaces of Linear Systems, visualizing the basic examples in R^2 and R^3, by means of concrete examples with linear equations of straight lines and planes. The concept of linear independence should be related to the number of independent variables needed to parameterize the solution object. This approach helps understanding the dimension and parameterizations of curves as solutions of non-linear systems, for instance as the intersection of surfaces, and the abstraction process to understand the solution spaces with dimension greater than 2 or 3. The combination of geometric and algebraic approaches is a way to modernize the curriculum with tangible meanings.

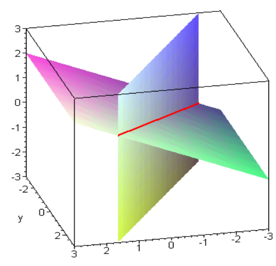

Figure 5: Visualizing a straight line as 1-dimensional solution space of a linear system.

- ✓ The study of Linear Transformations in the plane (and space) with Multiplication of Matrices, interpreting the dynamics of geometric transformations as an action of Group of Matrices on column-vectors, thus retrieving Klein's idea of Geometry with Transformation Groups. Such geometric approach is very effective to explain the rank-nullity theorem that can be illustrated at secondary school level to understand the loss of dimension by projected shadows of the objects, through manipulation of concrete objects.
- ✓ Relate the study of the spectrum of linear operators to the diagonal form of a matrix to investigate and understand the geometry of conic curves, and use this study to the applications in signal communication.
- ✓ Matrix algebra to understand the geometry of transformations, in particular the isometries of plane/space. Interpreting the matrix multiplication in the space of transformations as compositions of rigid motions produces immediate effect as student-projects on computer graphics, in particular the kinematics of robotics. Applicatons can be worked out at secondary school level through practical examples.
- ✓ The geometric interpretation of conjugation as change of variables implies the future understanding of the geometry of Lie groups.

The following Figure 6 illustrates the rank-nullity theorem and the geometric interpretation of conjugation.

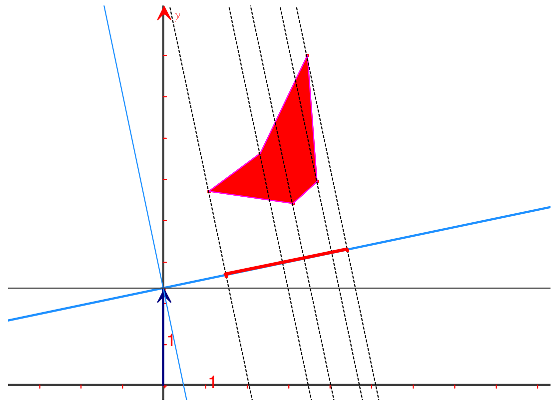

Figure 6: Change of coordinate system by means of conjugation by a translation composed with a rotation of axe; orthogonal projection as linear transformation with rank 1, geometric interpretation of nullity.

Other suggestions such as the geometric approaches to the study of determinant of matrices and of the concept of orientation are very significant to bring alternatives to teaching secondary school topics in mathematics that can lessen the gap between school instruction and the power of research discoveries.

CONCLUDING REMARKS: This note is a first attempt of the author to organize some ideas concerning the challenges to modernize the school curriculum that reflected the scientific advances of mathematics relevant to our lives. The Klein Project has brought an opportunity to this reflection, and the suggestions given in this note have been collected from the author's teaching experiences in undergraduate courses for sciences and technology courses, teacher preparation courses and outreach projects on continuing education of basic school teachers. Clearly the suggestions need to be ellaborated adequately to turn into accessible text by secondary school teachers, but any text to be produced in this Project should serve, quoting Klein ([2]), "to make it easier for you (…teacher) to acquire that ability to draw from the great body of knowledge a living stimulus for your teaching. … This book is designed solely as such a mental spur, not as a detailed handbook".

REFERENCES:

[1] Baldin, Y.Y., Slide Presentation "Some considerations about the advances of Mathematics in 20th century and the possible implications to modernize the school curriculum", Workshop Klein Project, Funchal, Madeira, October 2009.

[2] Barton, B., The Klein Project: A Living & Connected View of Mathematics for Teachers, An IMU-ICMI Collaboration: A Short Description, MSOR Connections Vol 8, No 4, November 2008- January 2009.