THREE APPROACHES TO KADISON-SINGER

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Abstract. These notes outline some of my ideas on approaches to the Kadison-Singer conjecture and are intended for the ARCC audience.

1. Introduction

Ever since J. Anderson's remarkable work, proving that the Kadison-Singer conjecture is equivalent to his paving conjecture, virtually all research on Kadison-Singer has focused on the paving conjecture. Even the many new equivalences that have arisen from the work of P. Casazza and his collaborators, such as the Feichtinger conjecture, are essentially at heart paving conjectures.

I would like to try and stimulate discussion about alternate approaches or perhaps learn from my colleagues why other approaches have been abandoned. In these notes I will discuss two alternate approaches to the Kadison-Singer conjecture that I would like to know more about, together with my thoughts on the paving conjecture. There is considerable overlap between my thoughts on the paving conjecture and a set of joint results to be submitted by P. Casazza, D. Edidin, D. Kalra and myself.

2. Crossed-Products

The original version of the Kadison-Singer conjecture asks whether or not pure states on the diagonal, $\mathcal{D}$, extend uniquely to pure states on $B(H)$. My basic idea here is to ask if we can parse this problem by choosing another C*-algebra in between, $\mathcal{D} \subset \mathcal{B} \subset B(H)$ and asking if pure states on $\mathcal{D}$ extend uniquely to pure states on $\mathcal{B}$ and if pure states on $\mathcal{B}$ extend uniquely to pure states on $B(H)$.

The algebra that I have in mind for an intermediate algebra is the crossed-product algebra. For these purposes, it will be convenient to regard, $H = \ell^2(\mathbb{Z})$ where $\mathbb{Z}$ denotes the group of integers, so that the diagonal matrices with respect to the natural basis is $D = \ell^\infty(\mathbb{Z})$. If we let B denote the bilateral shift, then the algebra that I have in mind for $\mathcal{B}$ is the C*-algebra generated by B and $\ell^\infty(\mathbb{Z})$. Note that conjugation by B translates elements of $\ell^\infty(\mathbb{Z})$ by one unit, which is the natural action of the group $\mathbb{Z}$ on $\ell^\infty(\mathbb{Z})$.

It is fairly well-known that the C*-algebra $\mathcal{B}$ is in fact, the C*-crossed product algebra, $\ell^\infty(\mathbb{Z}) \times \mathbb{Z}$.

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Problem 1. *Do pure states on $\mathcal{B}$ extend uniquely to pure states on $B(H)$?*

If this is true, then this would reduce the Kadison-Singer conjecture to:

Problem 2. *Do pure states on $\ell^\infty(\mathbb{Z})$ extend uniquely to pure states on the C*-crossed product, $\ell^\infty(\mathbb{Z}) \times \mathbb{Z}$?*

This suggests a whole area that I know nothing about, but perhaps others do. Given a discrete group, G, there is a natural action of G on $\ell^\infty(G)$ and we can ask:

Problem 3. *For which discrete groups, do pure states on $\ell^\infty(G)$ extend uniquely to pure states on $\ell^\infty(G) \times G$?*

Clearly, this last problem is some sort of variant of the Kadison-Singer conjecture.

Since $\ell^\infty(G)$ is an abelian C*-algebra, it might even be fruitful to look at an action of a group G on a compact, Hausdorff space X and ask:

Problem 4. *What properties of the action guarantees that pure states on $C(X)$ extend uniquely to pure states on $C(X) \times G$?*

3. The Stone-Cech Compactification

For consistency, I will stick to regarding the diagonal as $\ell^\infty(\mathbb{Z})$, although most of the things I say in this section don't need this.

Let $\beta\mathbb{Z}$ denote the Stone-Cech compactification of $\mathbb{Z}$. Since $\mathcal{D} = \ell^\infty(\mathbb{Z}) = C(\beta\mathbb{Z})$, we know that the pure state on $\mathcal{D}$ are given by point evaluations at arbitrary points in $\beta\mathbb{Z}$. In Kadison and Singer's original paper, they remark that the pure states corresponding to points in $\mathbb{Z}$, i.e., the vector inner products with the canonical basis vectors, all have unique pure state extensions. So the problem is with other pure state, i.e., the points in the so-called Corona set, $\beta\mathbb{Z} - \mathbb{Z}$.

We have tended to treat all of the corona points as "equal", but isn't it possible that if the Kadison-Singer conjecture is false, then some corona points will have unique extensions while others will not?

Thus, I ask:

Problem 5. *Is there a direct argument, that if one corona point has a unique pure state extension to B(H), then every corona point has a pure state extension?*

Even if we can't easily decide the answer to this last problem, I believe that if the Kadison-Singer conjecture is false, then it is likely that some corona points will have unique extensions and others will not. A quick glance through any book on the Stone-Cech compactifaction shows that the corona is very non-homogeneous. Thus, it is natural to look at various properties that points in the corona are known to have and try to guess which one's are likely to be "bad" for uniqueness of extension.

There are two types of points that my instincts tell me are likely candidates for counterexamples to uniqueness that I would like to bring to the groups attention.

The first type of points are the *P-points.* A point is a P-point if every G_δ containing the point is necessarily open. Caution-this is *not* equivalent to the notion of p-point from function theory!

Thus, if a continuous function vanishes at a P-point, then the zero set of the function is necessarily clopen. Equivalently, if ω is a P-point, then for every continuous function, $f, \{\gamma : f(\gamma) = f(\omega)\}$ is clopen. Hence, for every f, there is an open set of pure states that assign the same value to f.

It is not clear if P-points necessarily exist, but if we assume either the continuum hypothesis or Martin's axiom, then they not only exist, but the P-points and the non-P-points both form dense subsets of the corona. See, for example, [9].

Problem 6. *If $\omega \in \beta\mathbb{Z}$ is a P-point, then does the state given by evaluation at ω have a unique, pure state extension to B(H)?*

The other type of point that I would like to bring attention to are the *idempotent points.* One doesn't need any extra hypotheses to guarantee that these points exist. I used idempotent points in my work on Solel's conjecture to show some unusual examples related to spectral synthesis.

I'll outline the definition of idempotent points below.

These are best understood from the viewpoint of $\mathbb{Z}$ actions. Given a compact, Hausdorff space X and a homeomorphism, $h : X \to X$ we define an action of $\mathbb{Z}$ on X by setting, $n+x = h^{(n)}(x)$,, so that $0+x = x, n+(m+x) = (n+m)+x$. By the properties of the Stone-Cech compactifaction, for each fixed $x \in X$, the map $\mathbb{Z} \to X$ defined by $n \to n+x$ extends uniquely to a continuous map, $\beta\mathbb{Z} \to X$, which we write as $\omega \to \omega + x$. Note that given two points $\omega, \gamma \in \beta\mathbb{Z}$, we can write, $\omega + (\gamma + x)$.

This idea can be used to define a binary operation on $\beta\mathbb{Z}$. First, note that translation by one, is a homeomorphism of $\mathbb{Z}$ and hence extends uniquely to a continuous function, $h : \beta\mathbb{Z} \to \beta\mathbb{Z}$. Applying the above ideas to this map h, we see that for every pair of points $\omega, \gamma \in \beta\mathbb{Z}$ we can define a binary operation by $(\omega, \gamma) \to \omega + \gamma$.

The notation is deceptive, because this binary pairing is *not* commutative. However, it does satisfy that for any X and h, we have $\omega + (\gamma + x) = (\omega + \gamma) + x$.

Finally, a point ω is called *idempotent,* if $\omega + \omega = \omega$. Remarkably, using the compactness of the space $\beta\mathbb{Z}$ it can be shown that idempotent points exist [7].

Intuitively, if we think of points in $\beta\mathbb{Z}$ as "limits" then ω idempotent, implies that for any homeomorphism, h, we have

$$\lim_{n\to\omega}\lim_{k\to\omega} h^{(n+k)}(x) = \lim_{j\to\omega} h^{(j)}(x).$$

Another vital property of idempotent points is that if ω is idempotent, then the closure of the orbit of ω under the homeomorphism, $h\beta\mathbb{Z} \to \beta\mathbb{Z}$ induced by translation is a proper closed invariant subset of the corona. Moreover, there exists an h-equivariant retract of $\beta\mathbb{Z}$ onto this subset.

My vague idea is try and construct two different pure states via some sort of iterated limit process involving an idempotent point, who's restrictions to the diagonal are equal, because the point was idempotent.

Problem 7. *If $\omega \in \mathbb{Z}$ is an idempotent point, then does the pure state extension given by evaluation at ω possess non-unique extensions?*

There is another reason that I believe that an approach involving the Stone-Cech compactification might be fruitful. Many good mathematicians have struggled to construct an operator that fails paving to no avail. This approach is somewhat "transcendental". It could give a counterexample to the conjecture without giving any clues about how a non-pavable operator could be obtained. In fact, if the approach through P-points produced a counterexample, then we would be able to assert that non-pavable operators exist, assuming the continuum hypothesis! In which case they surely couldn't have been constructed.

4. Paving Conjectures

The only results that I have on the paving conjectures are contained in the joint notes [5]. However, we each have our own perspective and opinions, so I will state some of my views below, for which my co-authors should not be held responsible.

Let me begin with the familiar.

Given $A \subseteq \mathbb{I}$, where I is some index set, we let $Q_A \in B(\ell^2(\mathbb{I}))$ denote the diagonal projection defined by $Q_A = (q_{i,j}), q_{i,i} = 1, i \in A, q_{i,i} = 0, i \notin A$ and $q_{i,j} = 0, i \neq j$.

Definition 8. *An operator $T \in B(\ell^2(I))$ is said to have an (r, ϵ)-paving if there is a partition of I into r subsets $\{A_j\}_{j=1}^r$ such that $\|Q_{A_j} T Q_{A_j}\| \leq \epsilon$. A collection of operators $\mathcal{C}$ is said to be $(r\epsilon)$-pavable if each element of $\mathcal{C}$ has an (r, ϵ)-paving.*

Note that in this definition, I am *not* requiring that the diagonal entries of the operator be 0.

Some classes that will play a role are:

- $\mathcal{C}_\infty = \{T = (t_{i,j}) \in B(\ell^2(\mathbb{N})) : \|T\| \leq 1, t_{i,i} = 0 \forall i \in \mathbb{N}\}$,
- $\mathcal{C} = \cup_{n=2}^\infty \{T = (t_{i,j}) \in M_n : \|T\| \leq 1, t_{i,i} = 0, i = 1, ..., n\}$,
- $\mathcal{S}_\infty = \{T \in \mathcal{C}_\infty : T = T^*\}$,
- $\mathcal{S} = \{T \in \mathcal{C} : T = T^*\}$,
- $\mathcal{R}_\infty = \{T \in \mathcal{S}_\infty : T^2 = I\}$,
- $\mathcal{R} = \{T \in \mathcal{S} : T^2 = I\}$,
- $\mathcal{P}_{1/2}^\infty = \{T = (t_{i,j}) \in B(\ell^2(\mathbb{N})) : T = T^* = T^2, t_{i,i} = 1/2, \forall i \in \mathbb{N}\}$,

- $\mathcal{P}_{1/2} = \cup_{n=2}^{\infty}\{T = (t_{i,j}) \in M_n : T = T^* = T^2, t_{i,i} = 1/2, i = 1, ..., n\}$.

J. Anderson's[2] remarkable contribution follows.

Theorem 9 (Anderson)**.** *The following are equivalent:*

- *The Kadison-Singer conjecture is true,*
- *for each $T \in \mathcal{C}_\infty$, there exists (r, ϵ)(depending on T) $\epsilon < 1$, such that T is (r, ϵ)-pavable,*
- *there exists $(r, \epsilon), \epsilon < 1$, such that $\mathcal{C}_\infty$ is (r, ϵ)-pavablle,*
- *there exists $(r, \epsilon), \epsilon < 1$, such that $\mathcal{C}$ is (r, ϵ)-pavable,*
- *for each $T \in \mathcal{S}_\infty$, there exists $(r, \epsilon), \epsilon < 1$(depending on T), such that T is (r, ϵ)-pavable,*
- *there exists $(r, \epsilon), \epsilon < 1$, such that $\mathcal{S}_\infty$ is (r, ϵ)-pavable,*
- *there exists $(r, \epsilon), \epsilon < 1$, such that $\mathcal{S}$ is (r, ϵ)-pavable.*

Generally, when people talk about the paving conjecture they mean one of the above equivalences of the Kadison-Singer conjecture. Also, generally, when one looks at operators on an infinite dimensional space, it is enough to find (r, ϵ) depending on the operator, but for operators on finite dimensional spaces it is essential to have a uniform (r, ϵ). Finally, since $\mathcal{S}_\infty \subset \mathcal{C}_\infty$, people looking for counterexamples tend to study $\mathcal{C}_\infty$, while people trying to prove the theorem is true, study $\mathcal{S}_\infty$ or $\mathcal{S}$.

My first opinion, is that in either case, I want to study $\mathcal{S}_\infty$, because if a counterexample exists in $\mathcal{C}_\infty$ then one exists in $\mathcal{S}_\infty$ by the equivalences. Since we don't know which is the case, I prefer to study the set with the most structure and perhaps stumble across a proof while seeking a counterexample and vice versa.

In this spirit, we showed that the following smaller sets with "more structure" are sufficient for paving.

Theorem 10. [5] *The following are equivalent:*

- *The Kadison-Singer conjecture is true,*
- *for each $R \in \mathcal{R}_\infty$ there is a $(r, \epsilon), \epsilon < 1$(depending on R) such that R can be (r, ϵ)-paved,*
- *there exists $(r, \epsilon), \epsilon < 1$, such that every $R \in \mathcal{R}$ can be (r, ϵ)-paved,*
- *for each $P \in \mathcal{P}_{1/2}^{\infty}$ there is a $(r, \epsilon), \epsilon < 1$(depending on P) such that P can be (r, ϵ)-paved,*
- *there exists $(r, \epsilon), \epsilon < 1$, such that every $P \in \mathcal{P}_{1/2}$ can be paved.*

A sort of meta-corollary that goes with this latter result is that the frame based conjectures that are known to be equivalent to the Kadison-Singer result can be reduced to the case of uniform Parseval frames of redundancy 2. Similarly, for most harmonic analysis analogues of paving, it is enough to consider say subsets $E \subseteq [0, 1]$ of Lebesgue measure 1/2. I state one such equivalence, Casazza, et. al.[4] have shown that the *Feichtinger conjecture* is equivalent to Kadison-Singer.

Theorem 11. *The Feichtinger conjecture is true if and only if for each Parseval frame* $\{f_n\}_{n\in\mathbb{N}}$ *for a Hilbert space with* $\|f_n\|^2 = 1/2\ \forall n$ *there is a partition* $\{A_j\}_{j=1}^r$ *of* $\mathbb{N}$ *into r disjoint subsets(with r depending on the frame) such that for each* $j, \{f_n\}_{n\in A_j}$ *is a Riesz basis for the space that it spans.*

Finally, in [1] Akemann and Anderson introduce two paving conjectures, denoted *Conjecture A* and *Conjecture B.* It is known that Conjecture A implies Conjecture B and that Conjecture B implies Kadison-Singer, but it is not known if either of these implications can be reversed. Weaver[9] provides a set of counterexamples to Conjecture A. Thus, if these three statements were all equivalent then Weaver's counterexample would be the end of the story. However, it is generally believed that Conjecture A is strictly stronger than the Kadison-Singer conjecture.

In [5], we show that the Grammian projection matrix of any uniform, equiangular Parseval frame consisting of n vectors for a k dimensional space is a counterexample to Conjecture A, provided $n > 5k$. It is known that infinitely many such frames exist for arbitrarily large n and k. The significance of our new set of counterexamples is that by the results of J. Bourgain and L. Tzafriri [3], there exists $\epsilon < 1$, such that the family of self-adjoint, norm one, 0 diagonal matrices obtained from these frames is $(2, \epsilon)$-pavable.

Thus, these new examples drive an additional wedge between Conjecture A and Kadison-Singer.

I believe that Conjecture B is also false and that constructing a counterexample to this conjecture is within reach. In fact, I conjecture that we should be able to find counterexamples to Conjecture B, that belong to a "pavable" family of matrices.

Finally, Conjecture B is about paving projections with small diagonal. But our results show that Kadison-Singer is equivalent to paving projections with diagonal 1/2. This would also seem to put further distance between these Akemann-Anderson conjectures and the Kadison-Singer conjecture.

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