Paving Small Matrices and The Kadison-Singer Extension Problem
AIM Workshop Notes

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Part 1

Pavings
CHAPTER 1

Notation

\( \mathbb{M}_n = n \times n \) complex matrices
\( \mathbb{M}_n^0 = n \times n \) complex matrices with zero diagonal
\( \mathbb{M}_{n,sa} = n \times n \) selfadjoint complex matrices
\( \mathbb{M}_n^{0,sa} = n \times n \) selfadjoint complex matrices with zero diagonal
\( \mathbb{M}_n^{0,sym} = n \times n \) real symmetric matrices
\( \mathbb{M}_n^{0,sym} = n \times n \) real symmetric matrices with zero diagonal
\( \mathbb{M}_n^{++} = n \times n \) non-negative matrices
\( \mathbb{M}_n^{0,++} = n \times n \) non-negative matrices with zero diagonal
\( \mathbb{D}_n = n \times n \) diagonal matrices

If \( A \in \mathbb{M}_n \), define
\[
\alpha_k(A) = \min_{\text{diagonal projections } P_1 + \cdots + P_k = I_n} \max_{1 \leq j \leq k} ||P_jAP_j||
\]

If \( 0 \neq A \in \mathbb{M}_n \), define
\[
\tilde{\alpha}_k(A) = \frac{\alpha_k(A)}{\|A\|}.
\]

If \( S \subset \mathbb{M}_n \), define
\[
\tilde{\alpha}_k(S) = \sup_{0 \neq A \in S} \tilde{\alpha}_k(A).
\]
CHAPTER 2

2-Pavings

Theorem 2.1 (2-pavings).

<table>
<thead>
<tr>
<th>n</th>
<th>$\tilde{\alpha}_2(M^0_n)$</th>
<th>$\tilde{\alpha}<em>2(M^0</em>{n,sa})$</th>
<th>$\tilde{\alpha}<em>2(M^0</em>{n, sym})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>$\frac{\sqrt{3}}{3}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{\sqrt{3}}{3}$</td>
<td>$\frac{1}{2}$</td>
<td>$[0.5493, 0.5773]$</td>
</tr>
<tr>
<td>5</td>
<td>$0.5773$</td>
<td>$0.5000$</td>
<td>$[0.8944, 0.8944]$</td>
</tr>
</tbody>
</table>

1. Selfadjoint

Proposition 2.2 (3 × 3 selfadjoint). $\tilde{\alpha}_2(M^0_{3,sa}) = \frac{1}{\sqrt{3}} \approx 0.5773$.

Proof. Suppose $A \in M^0_{3,sa}$ with $\alpha_2(A) = 1$.

Then $|a|, |b|, |c| \geq 1$. By the Universal Selfadjoint 3-Identity (Lemma 7.1),

$$1 = \frac{|a|^2 + |b|^2 + |c|^2}{\|A\|^2} + \frac{2|Re(abc)|}{\|A\|^3} \geq \frac{3}{\|A\|^2}.$$

Thus, $\|A\| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_2(A) \leq \frac{1}{\sqrt{3}}$. This bound is attained by

$$A = \begin{bmatrix} 0 & 1 & i \\ 1 & 0 & 1 \\ -i & 1 & 0 \end{bmatrix}$$

because $\alpha_2(A) = 1$ and $\|A\| = \sqrt{3}$ by Corollary 7.2. \qed

Proposition 2.3 (4 × 4 selfadjoint). $\tilde{\alpha}_2(M^0_{4,sa}) = \frac{1}{\sqrt{3}}$.

Proof. Suppose $A \in M^0_{4,sa}$, with $\alpha_2(A) = 1$. Create a graph $G = (V, E)$ as follows: $V = \{1, 2, 3, 4\}$ and $(i, j) \in E$ if $|a_{ij}| < 1$. We have the following axioms:

1. $G11$ is not a subgraph of $G$. Otherwise, $A$ admits a 2-2 paving of norm $< 1$, violating the assumption $\alpha_2(A) = 1$.

2. For all $i$, the degree of $i$ is greater than 0. Otherwise, row $i$ of $A$ has three entries of absolute value $\geq 1 \Rightarrow \|A\| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{1}{\sqrt{3}}$.

3. By removing a vertex from $G$, one cannot arrive at $G4$. Otherwise, $A$ has a 3-compression of norm $\geq \sqrt{3} \Rightarrow \|A\| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_2(A) \leq \frac{1}{\sqrt{3}}$.
This exhausts all possible 4-graphs and hence proves the inequality.

**Proposition 2.4** (5 × 5 selfadjoint). Let \( \tilde{\alpha}_2(M_{5,sa}^0) = \frac{2}{\sqrt{5}} \approx 0.8944 \).

**Proof.** Suppose \( A \in M_{5,sa}^0 \) with \( \alpha_2(A) = 1 \). Create a graph \( G = (V,E) \) as follows: \( V = \{1, 2, 3, 4, 5\} \) and \((i, j) \in E\) if \( |a_{ij}| < 1 \). We may assume the following axiom:

1. For all \( i \), \( \deg(i) \geq 3 \). Otherwise, row \( i \) of \( A \) has at least two entries of absolute value \( \geq 1 \) \( \Rightarrow \|A\| \geq \sqrt{2} \Rightarrow \tilde{\alpha}_2(A) \leq \frac{1}{\sqrt{2}} \approx 0.7071 \).

This leaves graphs \( G_{50}, G_{51}, \) and \( G_{52} \).

**Case G50:** Only two 2-compressions have norm \( \geq 1 \), and they are disjoint. Without loss of generality, \( \|A_{12}\|, \|A_{34}\| \geq 1 \). We claim that every 3-compression has norm \( \geq 1 \). Indeed, \( \|A_{123}\| \geq \|A_{12}\| \geq 1, \|A_{345}\| \geq \|A_{34}\| \geq 1 \), and the remaining 3-compressions have norm \( \geq 1 \) because their complementary 2-compressions have norm \( < 1 \). It follows that \( \|A\| \geq \frac{\sqrt{2}}{2} \Rightarrow \tilde{\alpha}_2(A) \leq \frac{2}{\sqrt{5}} \).

**Case G51:** Only one 2-compression has norm \( \geq 1 \). Without loss of generality, \( \|A_{12}\| \geq 1 \). It follows that

\[
\|A\|^2 \geq \frac{1}{4} \|A\|_{HS}^2
= \frac{1}{4} \left[ \|A_{12}\|_{HS}^2 + \frac{1}{2} \sum_{1 \in B, 1 \not\in B} \|B\|_{HS}^2 + \frac{1}{2} \sum_{2 \in B, 1 \not\in B} \|B\|_{HS}^2 \right]
\geq \frac{1}{4} \left[ 2 + \frac{1}{2} \cdot 3 \cdot \frac{3}{2} + \frac{1}{2} \cdot 3 \cdot \frac{3}{2} \right] = \frac{13}{8}.
\]

Thus, \( \|A\| \geq \sqrt{\frac{13}{8}} \Rightarrow \tilde{\alpha}_2(A) \leq \sqrt{\frac{13}{8}} \approx 0.7845 \).

**Case G52:** Every 2-compression has norm \( < 1 \) \( \Rightarrow \) every 3-compression has norm \( \geq 1 \) \( \Rightarrow \|A\| \geq \frac{\sqrt{2}}{2} \Rightarrow \tilde{\alpha}_2(A) \leq \frac{2}{\sqrt{5}} \).

The matrix

\[
A = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & -1 \\
1 & 1 & 0 & -1 & 1 \\
1 & 1 & -1 & 0 & -1 \\
1 & -1 & 1 & -1 & 0
\end{bmatrix}
\]

shows that the inequality is sharp. The unimodular circulant

\[
B = \begin{bmatrix}
0 & e^{2\pi i/5} & e^{-\pi i/5} & e^{\pi i/5} & e^{-2\pi i/5} \\
e^{-2\pi i/5} & 0 & e^{2\pi i/5} & e^{-\pi i/5} & e^{\pi i/5} \\
e^{2\pi i/5} & e^{-\pi i/5} & 0 & e^{2\pi i/5} & e^{\pi i/5} \\
e^{-\pi i/5} & e^{\pi i/5} & e^{-2\pi i/5} & 0 & e^{2\pi i/5} \\
e^{2\pi i/5} & e^{-\pi i/5} & e^{\pi i/5} & e^{-2\pi i/5} & 0
\end{bmatrix}
\]

also works. Note: \( A \) and \( B \) are unitarily equivalent.

\( \square \)
Alternate Proof. Suppose $A \in \mathbb{M}_n^{sa}$, with $\alpha_2(A) = 1$.

(1) Assume that all 3-compressions of $A$ have norm $\geq 1$. Then $\tilde{\alpha}_2(A) \leq \frac{2}{\sqrt{5}}$ (see the previous proof).

(2) Assume that exactly one 3-compression, say $A_{345}$, has norm $< 1$, then $\|A_{12}\| \geq 1 \Rightarrow \tilde{\alpha}_2(A) \leq \sqrt{\frac{8}{13}}$ (see the previous proof).

(3) Assume that exactly two 3-compressions have norm $< 1$. We may assume that the complementary 2-compressions are disjoint. Otherwise, $\|A\| \geq \sqrt{2} \Rightarrow \tilde{\alpha}_2(A) \leq \frac{1}{\sqrt{3}}$. Without loss of generality, $\|A_{12}\|, \|A_{34}\| \geq 1$ and $\|A_{345}\|, \|A_{125}\| < 1$. This is a contradiction.

(4) Assume that more than two 3-compressions have norm $< 1$. Then their complementary 2-compressions cannot be disjoint. Thus, $\|A\| \geq \sqrt{2} \Rightarrow \tilde{\alpha}_2(A) \leq \frac{1}{\sqrt{2}}$.  

\[\square\]
2. Real Symmetric

PROPOSITION 2.5 ($3 \times 3$ real symmetric). $\tilde{\alpha}_2(M_{3,\text{sym}}^0) = \frac{1}{2}$.

PROOF. Suppose

$$A = \begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix} \in M_{3,\text{sym}}^0 \text{ with } \alpha_2(A) = 1.$$

Then $|a|, |b|, |c| \geq 1$. By the Universal Selfadjoint 3-Identity (Lemma 7.1),

$$1 = \frac{a^2 + b^2 + c^2}{\|A\|^2} + \frac{\sum |abc|}{\|A\|^3} \geq \frac{3}{\|A\|^2} + \frac{2}{\|A\|^3} \quad \text{which implies } \|A\| \geq 2,$$

hence $\tilde{\alpha}_2(A) \leq \frac{1}{2}$. This bound is attained by

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \in M_{3,\text{sym}}^0$$

since $\alpha_2(A) = 1$ and $\|A\| = 2$ by Corollary 7.2. □

Lemma 2.6. Let

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & d \\ e & 0 & f \end{bmatrix} \in M_{3,\text{sym}}.$$

If

$$\left\| \begin{bmatrix} 0 & d & e \\ d & 0 & f \\ e & f & 0 \end{bmatrix} \right\| \geq 1,$$

then $\|A\| \geq (9.75)^{1/4} \approx 1.767$.

PROOF. Let $x = [1 \quad 1 \quad 1]$ and

$$B = \begin{bmatrix} 0 & d & e \\ d & 0 & f \\ e & f & 0 \end{bmatrix}.$$

Then

$$A = \begin{bmatrix} 0 & x \\ x^* & B \end{bmatrix} \Rightarrow A^*A = \begin{bmatrix} xx^* & xB \\ B^*x & x^*x + B^*B \end{bmatrix}.$$

Thus

$$\|A\|^4 = \|A^*A\|^2 \geq \left\| \begin{bmatrix} xx^* & xB \end{bmatrix} \right\|^2$$

$$= 9 + (d + e)^2 + (d + f)^2 + (e + f)^2.$$

We claim that

$$(d + e)^2 + (d + f)^2 + (e + f)^2 \geq d^2 + e^2 + f^2.$$

Indeed, let $F(d, e, f) = (d + e)^2 + (d + f)^2 + (e + f)^2$ and $G(d, e, f) = d^2 + e^2 + f^2$. Using the Method of Lagrange Multipliers, we minimize $F(d, e, f)$ subject to the constraint $G(d, e, f) = r^2$. 
2. REAL SYMMETRIC

\[ 2(d + e) + 2(d + f) = 2\lambda d \]
\[ 2(d + e) + 2(e + f) = 2\lambda e \]
\[ 2(d + f) + 2(e + f) = 2\lambda f \]

\[ \Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \lambda \begin{bmatrix} d \\ e \\ f \end{bmatrix} \]

\[ \Rightarrow \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} x \\ x \\ x \end{bmatrix} \text{ or } \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} x + y \\ x - y \\ -2x \end{bmatrix}. \]

In the former case,
\[ 3x^2 = d^2 + e^2 + f^2 = r^2 \Rightarrow (d + e)^2 + (d + f)^2 + (e + f)^2 = 12x^2 = 4r^2. \]

In the later case,
\[ (x + y)^2 + (x - y)^2 + (-2x)^2 = d^2 + e^2 + f^2 = r^2 \]
\[ \Rightarrow (d + e)^2 + (d + f)^2 + (e + f)^2 = (2x)^2 + (-x + y)^2 + (-x - y)^2 = r^2. \]
Thus,
\[ r^2 \leq (d + e)^2 + (d + f)^2 + (e + f)^2 \leq 4r^2, \] which proves the claim. Now
\[ \|B\| \geq 1 \Rightarrow \|B\|^2_{HS} \geq 1.5 \Rightarrow d^2 + e^2 + f^2 \geq 0.75. \]

Hence, \( \|A\|^4 \geq 9.75, \) which proves the lemma.

\[ \Box \]

**Proposition 2.7 (4 × 4 real symmetric).** \( \tilde{\alpha}_2(M^0_{4,\text{sym}}) \in [0.5493, 0.5773]. \)

**Proof.** Suppose \( A \in M^0_{4,\text{sym}}, \) with \( \alpha_2(A) = 1. \) Create a graph \( G = (V, E) \) as follows: \( V = \{1, 2, 3, 4\} \) and \( (i, j) \in E \) if \( |a_{ij}| < 1. \) We have the following axioms:

1. \( G11 \) is not a subgraph of \( G. \) Otherwise, \( A \) admits a 2-2 paving of norm \( < 1, \) violating the assumption \( \alpha_2(A) = 1. \)
2. By removing a vertex from \( G, \) one cannot arrive at \( G4. \) Otherwise, \( A \) has a 3-compression of norm \( \geq 2 \) \( \Rightarrow \|A\| \geq 2 \Rightarrow \tilde{\alpha}_2(A) \leq \frac{1}{2}. \)

This leaves only graph \( G12. \) Thus,
\[
A = \begin{bmatrix} 0 & a & b & c \\ a & 0 & d & e \\ b & d & 0 & f \\ c & e & f & 0 \end{bmatrix},
\]

where \( |a|, |b|, |c| \geq 1, |d|, |e|, |f| < 1, \) and
\[
\begin{bmatrix} 0 & d & e \\ d & 0 & f \\ e & f & 0 \end{bmatrix} \geq 1.
\]

**Lower bound:**
\[
A = \begin{bmatrix} 0 & 1 & -0.3946 & 0.6854 \\ 1 & 0 & -0.3946 & 0.6854 \\ -0.3946 & 0 & -0.3986 & 0 \\ 0.6854 & -0.3986 & 0 & -0.3946 \end{bmatrix}.
\]

\[ \Box \]
CHAPTER 3

3-Pavings

In 1987 the 3-paving problem was posed: whether or not 3-pavings suffice for Anderson’s Paving Conjecture and hence for Kadison-Singer. To date we have heard of no refutation to this. Recall also the 3/3-challenge from then: whether or not \( \hat{\alpha}_3(M_n^0) \leq \frac{2}{3} \), which the following table refutes.

**Theorem 3.1 (3-pavings).**

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \hat{\alpha}_3(M_n^0) )</th>
<th>( \hat{\alpha}<em>3(M</em>{n,sa}^0) )</th>
<th>( \hat{\alpha}<em>3(M</em>{n,+}^0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>( \frac{1}{1+\sqrt{5}} ) 0.6180</td>
<td>( \frac{1}{\sqrt{3}} ) 0.5773</td>
<td>( \kappa ) 0.5550</td>
</tr>
<tr>
<td>5</td>
<td>&quot;</td>
<td>&quot;</td>
<td>( [\kappa, \frac{2}{1+\sqrt{5}}] ) [0.5550, 0.6180]</td>
</tr>
<tr>
<td>6</td>
<td>( \frac{1}{\sqrt{3}} ) 0.7071</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
<tr>
<td>7</td>
<td>( [?, 1] ) [0.8231, 1]</td>
<td>( \frac{2}{3}, \frac{2}{\sqrt{3}} ) [0.6667, 0.7559]</td>
<td>( [\kappa, \frac{2}{3}] ) [0.5550, 0.6667]</td>
</tr>
<tr>
<td>8</td>
<td>( [?, 1] ) [0.8231, 1]</td>
<td>( \frac{2}{3}, \frac{2}{\sqrt{3}} ) [0.6667, 0.8944]</td>
<td>&quot;</td>
</tr>
<tr>
<td>10</td>
<td>&quot;</td>
<td>( \frac{2}{3} ) [0.7454, 1]</td>
<td>&quot;</td>
</tr>
</tbody>
</table>

where

\[
\kappa = \sqrt{\frac{3}{5 + 2\sqrt{7}\cos(\tan^{-1}(3\sqrt{3}/3))}}
\]

**boldface signifies what we feel are the most interesting facts, "?" signifies a lack of a closed form, and "ditto from above".**
1. **General**

**Lemma 3.2.** Let

\[ A = \begin{bmatrix} r_1 e^{i\theta_1} & r_2 e^{i\theta_2} & r_3 e^{i\theta_3} \end{bmatrix} \in \mathbb{M}_2. \]

Then there exist unitaries \( U, V \in \mathbb{D}_2 \) such that

\[ UAV = \begin{bmatrix} r_1 & r_2 & 0 \\
0 & r_3 & 0 \end{bmatrix}. \]

**Proof.** Let

\[ U = \begin{bmatrix} e^{-i\theta_2} & 0 \\
0 & e^{-i\theta_3} \end{bmatrix}, \quad V = \begin{bmatrix} e^{i(\theta_2 - \theta_1)} & 0 \\
0 & 1 \end{bmatrix}. \]

\[ \square \]

**Corollary 3.3.** Let

\[ A = \begin{bmatrix} a & b \\
0 & c \end{bmatrix} \in \mathbb{M}_2. \]

If \(|a|, |b|, |c| \geq 1\), then \( \|A\| \geq \frac{1 + \sqrt{5}}{2} \).

**Proof.** By the previous lemma,

\[ \|A\| = \left\| \begin{bmatrix} a & b \\
0 & c \end{bmatrix} \right\| \geq \left\| \begin{bmatrix} 1 & 1 \\
0 & 1 \end{bmatrix} \right\| = \frac{1 + \sqrt{5}}{2}. \]

\[ \square \]

**Proposition 3.4 (4 \times 4 general).** \( \tilde{\alpha}_3(\mathbb{M}_4) = \frac{2}{1 + \sqrt{5}} \approx 0.6180. \)

**Proof.** Let

\[ A = \begin{bmatrix} 0 & 1 & 1 & -\frac{2}{1 + \sqrt{5}} \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{M}_4. \]

Then \( \tilde{\alpha}_3(A) = \frac{2}{1 + \sqrt{5}} \) (\( \alpha_3(A) = 1 \) and \( \|A\| = \frac{1 + \sqrt{5}}{2} \) by applying to the upper-right 3 \times 3 corner either Parrott’s Completion Lemma with Formula, or factoring the characteristic polynomial of the square of its absolute value, or Matlab).

Now suppose \( A \in \mathbb{M}_4^0 \), with \( \alpha_3(A) = 1 \). Create a digraph \( D = (V, E) \) as follows: \( V = \{1, 2, 3, 4\} \) and \( (i, j) \in E \) if \( |a_{ij}| \geq 1 \). We may assume the following axioms:

(1) For all \( i \neq j \), either \( (i, j) \in E \) or \( (j, i) \in E \). Otherwise \( A \) admits a 1-1-2 paving of norm \(< 1\), violating the assumption \( \alpha_3(A) = 1 \).

(2) For all \( i \), the in-degree of \( i \) and the out-degree of \( i \) are less than 3. Otherwise, \( \|A\| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{1}{\sqrt{3}} \approx 0.5774 \).

This leaves only digraphs \( D_{149}, D_{185}, D_{186}, \) and \( D_{218} \) as labeled in [1]. Now each of these digraphs has \( D_{12} \) as a subgraph [ibid.]. Thus, \( \|A\| \geq \frac{1 + \sqrt{5}}{2} \) (Corollary 3.3) \( \Rightarrow \tilde{\alpha}_3(A) \leq \frac{2}{1 + \sqrt{5}}. \)

\[ \square \]
PROPOSITION 3.5 (5 × 5 general). \( \tilde{\alpha}_3(M^0_5) = \frac{2}{1 + \sqrt{5}} \approx 0.6180. \)

PROOF. Clearly,

\[ \tilde{\alpha}_3(M^0_5) \geq \tilde{\alpha}_3(M^0_4) = \frac{2}{1 + \sqrt{5}}. \]

Now let \( A \in M^0_5 \), with \( \alpha_3(A) = 1 \). Construct a graph \( G = (V, E) \) as follows:

1. \( G \) is not a subgraph of \( G \). Otherwise, \( G \) has a 1-2-2 paving of norm < 1, violating the fact that \( \alpha_3(A) = 1 \).

2. By removing a vertex from \( G \) one cannot arrive at \( G^8 \). Otherwise, there exists a 4-compression \( B \) of \( A \) such that \( \alpha_3(B) \geq 1 \). Since \( \alpha_3(M^0_4) = \frac{2}{1 + \sqrt{5}} \),

this would imply \( \|B\| \geq \frac{1 + \sqrt{5}}{2} \Rightarrow \|A\| \geq \frac{1 + \sqrt{5}}{2} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{2}{1 + \sqrt{5}} \).

This leaves \( G^{23} \). After permuting indices, we may assume that

\[
A = \begin{bmatrix}
0 & s_{12} & s_{13} & b_{14} & b_{15} \\
s_{21} & 0 & s_{23} & b_{24} & b_{25} \\
s_{31} & s_{32} & 0 & b_{34} & b_{35} \\
b_{41} & b_{42} & b_{43} & 0 & b_{45} \\
b_{51} & b_{52} & b_{53} & b_{54} & 0
\end{bmatrix},
\]

where \( s_{ij} < 1 \) and \( \max\{|b_{ij}|, |b_{ji}|\} \geq 1 \) for all \( i \neq j \). Permuting the indices 4 and 5, if necessary, we may assume \( |b_{15}| \geq 1 \). If \( b_{51}, b_{52}, b_{53} \) have magnitude \( \geq 1 \), then \( \|A\| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{1 + \sqrt{5}}{2} \).

Thus, we may assume that one of them has magnitude < 1 \( \Rightarrow \) either \( b_{15}, b_{25}, \) or \( b_{35} \) has magnitude \( \geq 1 \). Permuting the indices 1, 2, and 3, if necessary, we may assume \( |b_{35}| \geq 1 \). If \( |b_{34}| \geq 1 \), then

\[ \|A\| \geq \| \begin{bmatrix} b_{34} & b_{35} \\ 0 & b_{45} \end{bmatrix} \| \geq \frac{1 + \sqrt{5}}{2}. \]

Likewise, if \( |b_{43}| \geq 1 \), then

\[ \|A\| \geq \| \begin{bmatrix} 0 & b_{35} \\ b_{43} & b_{45} \end{bmatrix} \| \geq \frac{1 + \sqrt{5}}{2}. \]

It follows that \( \tilde{\alpha}_3(A) \leq \frac{2}{1 + \sqrt{5}} \). \( \square \)

PROPOSITION 3.6 (6 × 6 general). \( \tilde{\alpha}_3(M^0_6) = \frac{1}{\sqrt{2}} \approx 0.7071 \).

PROOF. Construct a graph \( G = (V, E) \) as follows: \( V = \{1, 2, 3, 4, 5, 6\} \) and \( (i, j) \in E \) if \( |a_{ij}|, |a_{ji}| < 1 \). We may assume the following axioms:

1. \( G \) is not a subgraph of \( G \). Otherwise \( A \) would have a 2-2-2 paving of norm < 1, violating the fact that \( \alpha_3(A) = 1 \).

2. By removing vertices from \( G \), one cannot arrive at \( G^8 \). Otherwise \( A \) would have a 4-compression \( B \) such that \( \alpha_3(B) \geq 1 \). Since \( \alpha_3(M^0_4) = \frac{2}{1 + \sqrt{5}} \),

this would imply \( \|B\| \geq \frac{1 + \sqrt{5}}{2} \Rightarrow \|A\| \geq \frac{1 + \sqrt{5}}{2} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{2}{1 + \sqrt{5}} \).

3. For all vertices \( i, \deg(i) \geq 3 \). Otherwise, if \( \deg(i) \leq 2 \), then either row \( i \) or column \( i \) of \( A \) would have at least two entries of magnitude \( \geq 1 \Rightarrow \|A\| \geq \sqrt{2} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{1}{\sqrt{2}} \).
This eliminates all graphs. Now let
\[
A = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 1 \\
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 \\
-\frac{1}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
\frac{1}{2} & 1 & -\frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} & 0 \\
\end{bmatrix} \in M_6^0.
\]
Then \( \alpha_3(A) = 1 \) and \( A^\ast A = 2I \). \( \square \)

**Proposition 3.7 (7 \times 7 general).** \( \tilde{\alpha}_3(M_7^0) \in [0.8231, 1) \).

**Proof.** The following matrix was discovered by searching among 7 \times 7 unitary circulants for bad pavers. The starting point for the search was a 7 \times 7 unitary circulant with the eigenvalue distribution \((1, e^{\pi i/3}, e^{-\pi i/3}, i, -i, -1, -1)\).

\[
A = \begin{bmatrix}
a & b & c & d & e & f \\
f & 0 & a & b & c & d \\
e & f & 0 & a & b & c \\
d & e & f & 0 & a & b \\
c & d & e & f & 0 & a \\
a & b & c & d & e & f
\end{bmatrix},
\]

where
\[
a = -0.19104830537481 - 0.1857143276728i \\
b = 0.03404378754044 + 0.00110165928527i \\
c = -0.13926357252448 + 0.42165365488402i \\
d = 0.21474405201775 - 0.42217403069332i \\
e = -0.2833739310887 - 0.48101315713848i \\
f = 0.29151538363540 - 0.33115367910212i.
\]

Then \( \alpha_3(A) = 0.82305627367962 \) and \( A^\ast A = I \), i.e. \( \tilde{\alpha}_3(A) = 0.82305627367962 \).

It remains to show that \( \tilde{\alpha}_3(M_7^0) \neq 1 \). To that end, let \( A \in M_7^0 \), with \( \alpha_3(A) = 1 \). If every 3-compression of \( A \) has norm \( \geq 1 \), then \( \|A\| > 1 \) (Corollary 7.10). If, on the other hand, some 3-compression of \( A \) has norm \( < 1 \), then the complementary 4-compression \( B \) satisfies \( \alpha_2(B) \geq 1 \). In particular, every 2-2 paving of \( B \) has norm \( \geq 1 \). By Lemma 7.11, we may assume that
\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & * & * \\
0 & a & 0 & 0 & 0 & 0 \\
0 & 0 & b & 0 & 0 & 0 \\
0 & c & 0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 & * & * \\
* & 0 & 0 & 0 & * & 0 \\
* & 0 & 0 & 0 & * & 0
\end{bmatrix},
\]

where \(|a| = |b| = |c| = 1 \) and \( \|A_{567}\| < 1 \). Since \( \|A_{12}\| = \|A_{35}\| = 0, \|A_{467}\| = 1 \Rightarrow \|A_{67}\| = 1 \Rightarrow \|A_{567}\| = 1 \), a contradiction. \( \square \)
2. Self-adjoint

**Proposition 3.8 (4 × 4 self-adjoint).** \( \tilde{\alpha}_3(M_{4, sa}^0) = \frac{1}{\sqrt{3}} \approx 0.5773. \)

**Proof.** Suppose \( A \in M_{4, sa}^0 \), with \( \alpha_3(A) = 1 \). Then \( |a_{ij}| \geq 1 \) for all \( i \neq j \). Thus, \( ||A|| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{1}{\sqrt{3}} \). Now let

\[
A = \begin{bmatrix}
0 & i & 1 & 1 \\
-i & 0 & 1 & -1 \\
1 & 1 & 0 & i \\
1 & -1 & -i & 0
\end{bmatrix} \in M_{4, sa}^0.
\]

Then \( \tilde{\alpha}_3(A) = \frac{1}{\sqrt{3}} (\alpha_3(A) = 1 \text{ and } A^*A = 3I) \). \( \square \)

**Proposition 3.9 (5 × 5 self-adjoint).** \( \tilde{\alpha}_3(M_{5, sa}^0) = \frac{1}{\sqrt{3}} \).

**Proof.** Clearly,

\[
\tilde{\alpha}_3(M_{5, sa}^0) \geq \tilde{\alpha}_3(M_{4, sa}^0) = \frac{1}{\sqrt{3}}.
\]

Now let \( A \in M_{5, sa}^0 \), with \( \alpha_3(A) = 1 \). Construct a graph \( G = (V, E) \) as follows: 
\( V = \{1, 2, 3, 4, 5\} \) and \( (i, j) \in E \) if \( |a_{ij}| < 1 \) \( (\Rightarrow |a_{ji}| < 1) \). We may assume the following axioms:

1. **G11** is not a subgraph of \( G \). Otherwise, \( A \) would have a 1-2-2 paving of norm \( < 1 \), violating the assumption \( \alpha_3(A) = 1 \).
2. By removing a vertex from \( G \), one cannot arrive at **G8**. Otherwise, \( A \) would have a 4-compression \( B \) such that \( \alpha_3(B) \geq 1 \). Since \( \tilde{\alpha}_3(M_{4, sa}^0) = \frac{1}{\sqrt{3}} \), this would imply \( ||B|| \geq \sqrt{3} \Rightarrow ||A|| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{1}{\sqrt{3}} \).
3. For every vertex \( i, \text{deg}(i) \geq 2 \). Otherwise, if \( \text{deg}(i) \leq 1 \), then row \( i \) of \( A \) has at least three entries of magnitude \( \geq 1 \Rightarrow ||A|| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{1}{\sqrt{3}} \). This eliminates all graphs. \( \square \)

**Proposition 3.10 (6 × 6 self-adjoint).** \( \tilde{\alpha}_3(M_{6, sa}^0) = \frac{1}{\sqrt{3}} \).

**Proof.** Clearly,

\[
\tilde{\alpha}_3(M_{6, sa}^0) \geq \tilde{\alpha}_3(M_{5, sa}^0) = \frac{1}{\sqrt{3}}.
\]

Now let \( A \in M_{6, sa}^0 \), with \( \alpha_3(A) = 1 \). Construct a graph \( G = (V, E) \) as follows: 
\( V = \{1, 2, 3, 4, 5, 6\} \) and \( (i, j) \in E \) if \( |a_{ij}| < 1 \) \( (\Rightarrow |a_{ji}| < 1) \). We may assume the following axioms:

1. **G61** is not a subgraph of \( G \). Otherwise, \( A \) would have a 2-2-2 paving of norm \( < 1 \), violating the assumption \( \alpha_3(A) = 1 \).
2. By removing a vertex from \( G \), one cannot arrive at **G8**. Otherwise, \( A \) would have a 4-compression \( B \) such that \( \alpha_3(B) \geq 1 \). Since \( \tilde{\alpha}_3(M_{4, sa}^0) = \frac{1}{\sqrt{3}} \), this would imply \( ||B|| \geq \sqrt{3} \Rightarrow ||A|| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{1}{\sqrt{3}} \).
3. For every vertex \( i, \text{deg}(i) \geq 3 \). Otherwise, if \( \text{deg}(i) \leq 2 \), then row \( i \) of \( A \) has at least three entries of magnitude \( \geq 1 \Rightarrow ||A|| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{1}{\sqrt{3}} \). This eliminates all graphs. \( \square \)
Preliminaries for $7 \times 7$ Selfadjoints

Notation: $F = [1 - \delta_{ij}] \in \mathbb{M}^0_{n,sa}$ (the “fat” operator)

**Lemma 3.11.** Let $0 \neq A \in \mathbb{M}^0_{n,sa}$. Then the following are equivalent:

i. $\|A\|^2 = \frac{n-1}{n}\|A\|^2_{HS}$.

ii. There exists a nonzero $\alpha \in \mathbb{R}$ such that 

$$
\sigma(\alpha^{-1}A) = \left(1, -\frac{1}{n-1}, -\frac{1}{n-1}, ..., -\frac{1}{n-1}\right).
$$

iii. There exists a diagonal unitary $U \in \mathbb{D}_n$ and a nonzero $\beta \in \mathbb{R}$ such that 

$$
U^*A U = \beta F.
$$

**Proof.** (i $\Leftrightarrow$ ii): We have seen that $\|A\|^2 = \frac{n-1}{n}\|A\|^2_{HS}$ if and only if 

$$
\sigma(A) = \pm \|A\| \left(1, -\frac{1}{n-1}, -\frac{1}{n-1}, ..., -\frac{1}{n-1}\right).
$$

(ii $\Leftrightarrow$ iii): Set $\tilde{A} = \alpha^{-1}A$. If $\sigma(\tilde{A}) = \left(1, -\frac{1}{n-1}, -\frac{1}{n-1}, ..., -\frac{1}{n-1}\right)$, then there exists a unitary $U \in \mathbb{M}_n$ such that 

$$
\tilde{A} = V \begin{bmatrix}
1 & 0 & 0 & \hdots & 0 \\
0 & \frac{1}{n-1} & 0 & \hdots & 0 \\
0 & 0 & -\frac{1}{n-1} & \hdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \hdots & -\frac{1}{n-1}
\end{bmatrix} V^*.
$$

Letting $v$ stand for the first column of $V$, we have that 

$$
A = \frac{n}{n-1} \|v\|^2 - \frac{1}{n-1} I = \left[\frac{n}{n-1}v_i v_j - \frac{1}{n-1}\delta_{ij}\right].
$$

Since $\tilde{A} \in \mathbb{M}^0_{n,sa}$,

$$
\frac{n}{n-1}|v_i|^2 - \frac{1}{n-1} = 0 \Rightarrow v_i = \frac{1}{\sqrt{n}} e^{i\theta_i}
$$

for some $\theta_i \in \mathbb{R}$. It follows that 

$$
\tilde{A} = \frac{1}{n-1} \left[e^{i(\theta_i - \theta_j)} - \delta_{ij}\right] = \frac{1}{n-1} UFU^*,
$$

where 

$$
U = \text{diag}(e^{i\theta_1}, e^{i\theta_2}, ..., e^{i\theta_n}) \in \mathbb{D}_n.
$$

Thus, $U^*AU = \beta F$, where $\beta = \frac{n}{n-1}$. (iii $\Rightarrow$ ii): Clearly 

$$
F = nE - I,
$$

where all the off-diagonal entries of $E \in \mathbb{M}_n$ equal $\frac{1}{n}$. Since $E$ is a rank-one projection, 

$$
\sigma(F) = (n-1, -1, -1, ..., -1).
$$

The result follows. \qed
LEMMA 3.12. Let 0 ≠ A ∈ \( M^0_{n,sa} \). Fix k ≥ 3 and assume \( \|B\|^2 = \frac{k-1}{k} \|B\|^2_{HS} \) for all k-compressions B of A. Then there exists a diagonal unitary \( U \in \mathbb{D}_n \) and an \( \alpha > 0 \) such that
\[ U^* AU = \alpha S, \]
where all the off-diagonal entries of \( S \in M^0_{n,sa} \) equal \( \pm 1 \).

PROOF. Let B be a k-compression of A. By Lemma 3.11, all the off-diagonal entries of B have the same modulus. It follows that all the off-diagonal entries of A have the same modulus, say \( \alpha \) (here we use \( k \geq 3 \)). Set \( C = \alpha^{-1} A \). Then all the off-diagonal entries of C have modulus 1, and \( \|B\|^2 = \frac{k-1}{k} \|B\|^2_{HS} \) for all k-compressions B of C. We claim that \( c_{rs}c_{st} = \pm c_{rt} \) for all \( r < s < t \). Indeed, this follows from Lemma 3.11 applied to any k-compression B of C containing \( r, s, \) and \( t \) (again we use \( k \geq 3 \)). Now let \( \phi_1, \phi_2, \ldots, \phi_{n-1} \in \mathbb{R} \) be such that \( c_{i,i+1} = e^{i\phi_i} \), \( i = 1, 2, \ldots, n-1 \). For \( j = 1, 2, \ldots, n \), define \( \theta_j = -\sum_{i=1}^{j-1} \phi_i \). We claim that
\[ c_{rs} = \pm e^{i(\theta_r - \theta_s)}, \quad r < s. \]
Indeed,
\[ c_{rs} = \pm c_{r,r+1}c_{r+1,r+2} \cdots c_{s-1,s} = \pm e^{i\phi_r}e^{i\phi_{r+1}} \cdots e^{i\phi_{s-1}} \]
\[ = \pm e^{i\sum_{i=r}^{s-1} \phi_i} = \pm e^{i(\sum_{i=1}^{s} \phi_i - \sum_{i=1}^{r-1} \phi_i)} = \pm e^{i(\theta_s - \theta_r)}. \]
Setting
\[ U = \text{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}) \in \mathbb{D}_n, \]
we have that \( U^* C U = S \in M^0_{n,sa} \), where all the off-diagonal entries of S are \( \pm 1 \). \( \Box \)

PROPOSITION 3.13 (7 × 7 selfadjoint). \( \tilde{\alpha}_3(M^0_{7,sa}) \in \left[ \frac{2}{3}, \frac{2}{\sqrt{7}} \right] \approx [0.6667, 0.7559]. \)

PROOF. Let
\[ A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 0 & -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 0 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 0 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 0 & 1 \\ 1 & -1 & -1 & 1 & 1 & 0 & 1 \end{bmatrix} \in M^0_{7,sa}. \]

Then \( \tilde{\alpha}_3(A) = \frac{2}{3} \) (\( \alpha_3(A) = 2 \) and \( \|A\| = 3 \)). Thus, \( \tilde{\alpha}_3(M^0_{7,sa}) \geq \frac{2}{3} \). Now let \( A \in M^0_{7,sa} \), with \( \alpha_3(A) = 1 \).

If every 3-compression B of selfadjoint A has norm \( \geq 1 \), then \( \|B\|^2 \geq \frac{2}{3} \|B\|^2 \) by selfadjointness using Proposition 7.5 (\( p = 2, n = 3 \)).

General identity: \( \sum_{B} \|B\|^2_{HS} = 5\|A\|^2_{HS} \) by a counting argument.

From general selfadjoint trace zero inequality for odd rank: \( \|A\|^2_{HS} \leq 6\|A\|^2 \) by Corollary 7.4 (\( n = 7 \)). Thus
\[ 35 \leq \sum_{B} \|B\|^2 \leq \frac{2}{3} \sum_{B} \|B\|^2_{HS} = \frac{10}{3}\|A\|^2_{HS} \leq 20\|A\|^2 \]
and hence \( \|A\| \geq \frac{\sqrt{7}}{2} \) ⇒ \( \tilde{\alpha}_3(A) \leq \frac{2}{\sqrt{7}} \).

That \( \|A\| \geq \frac{\sqrt{7}}{2} \) is a special case of Corollary 7.6 (\( n = 7, k = 3 \)), so the above internal proof of this can alternatively be referenced.
If, on the other hand, some 3-compression of $A$ has norm $< 1$, then the complementary 4-compression $B$ satisfies $\alpha_2(B) \geq 1$. Since $\tilde{\alpha}_2(M_{4,sa}^0) = \frac{1}{\sqrt{3}}$, $\|B\| \geq \sqrt{3}$ implies $\|A\| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{3}{\sqrt{3}} < \frac{2}{\sqrt{7}}$.

Now assume $\alpha_3(A) = 1$ and $\|A\| = \frac{2\sqrt{7}}{3}$. By the previous discussion, every 3-compression $B$ of $A$ has norm $\geq 1$. Thus

\[ 35 \leq \sum_B \|B\|^2 \leq \frac{2}{3} \sum_B \|B\|_{HS}^2 = \frac{10}{3} \|A\|_{HS}^2 \leq 20 \|A\|^2 = 35. \]

It follows that $\|B\|^2 = \frac{2}{3} \|B\|_{HS}^2$ for all 3-compressions $B$ of $A$. By Lemma 5.6, there exists a diagonal unitary $U \in \mathbb{D}_n$ and an $\alpha > 0$ such that $U^*AU = \alpha S$, where all the off-diagonal entries of $S \in M_{n,sa}^0$ are $\pm 1$. Searching exhaustively among all such $S$, we see that $\tilde{\alpha}_3(A) \leq \frac{2}{\sqrt{7}} < \frac{2}{\sqrt{3}}$, a contradiction.

\[ \text{□} \]

**Proposition 3.14 (8 × 8 selfadjoint).** $\tilde{\alpha}_3(M_{8,sa}^0) \in \left[\frac{2}{3}, \frac{2\sqrt{7}}{3}\right] \approx [0.6667, 0.8944].$

**Proof.** Clearly,

\[ \tilde{\alpha}_3(M_{8,sa}^0) \geq \tilde{\alpha}_3(M_{7,sa}^0) \geq \frac{2}{3}. \]

Now let $A \in M_{8,sa}^0$ with $\alpha_3(A) = 1$. If every 3-compression of $A$ has norm $\geq 1$, then $\|A\| \geq \frac{2\sqrt{7}}{3}$ (by proof of 3.13 every 7-compression has norm $\geq \frac{2\sqrt{7}}{3}$) $\Rightarrow \tilde{\alpha}_3(A) \leq \frac{3}{\sqrt{7}} < \frac{2}{\sqrt{3}}$. If, on the other hand, some 3-compression of $A$ has norm $< 1$, then the complementary 5-compression $B$ satisfies $\alpha_2(B) \geq 1$. Since $\tilde{\alpha}_2(M_{5,sa}^0) = \frac{2}{\sqrt{5}}$, $\|B\| \geq \frac{2\sqrt{7}}{3} \Rightarrow \|A\| \geq \frac{2\sqrt{7}}{3} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{2}{\sqrt{5}}$. \[ \text{□} \]

**Proposition 3.15 (10 × 10 selfadjoint).** $\tilde{\alpha}_3(M_{10,sa}^0) \in \left[\frac{2\sqrt{5}}{3}, 1\right] \approx [0.7454, 1].$

**Proof.** Let

\[ A = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 \\
1 & 1 & 0 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & 0 & 1 & 1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & 0 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & 1 & 0 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 0 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 0 \\
1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 0 \\
\end{bmatrix} \in M_{10,sa}^0. \]

Then $\tilde{\alpha}_3(A) = \frac{2\sqrt{5}}{3}$ ($\alpha_3(A) = \sqrt{5}$ and $A^*A = 9I$). \[ \text{□} \]

Remark: $A$ is a conference matrix.
3. Nonnegative

**Lemma 3.16.** Let $A \in M_{4,+}^{0}$. If $\alpha_3(A) = 1$ and a row or column of $A$ has three entries $\geq 1$, then $\|A\| \geq 2$. This inequality is sharp.

**Proof.** We may assume the first row of $A$ has three entries $\geq 1$. Then

$$\|A\| \geq \left\| \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & b_{23} & b_{24} \\ 0 & b_{32} & 0 & b_{34} \\ 0 & b_{42} & b_{43} & 0 \end{bmatrix} \right\|,$$

where $\max\{b_{ij}, b_{ji}\} \geq 1$ for all $i \neq j$. Since

$$\min \left\{ \left\| \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & \delta_{23} & \delta_{24} \\ 0 & 1 - \delta_{23} & 0 & \delta_{34} \\ 0 & 1 - \delta_{24} & 1 - \delta_{34} & 0 \end{bmatrix} \right\| : \delta_{23}, \delta_{24}, \delta_{34} \in \{0, 1\} \right\} = 2,$$

we have that $\|A\| \geq 2$. A sharp example is furnished by the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

**Proposition 3.17 (4 × 4 nonnegative).** $\tilde{\alpha}_3(M_{4,+}^{0}) = \kappa \approx 0.5550.$

**Proof.** Suppose $A \in M_{4,+}^{0}$, with $\alpha_3(A) = 1$. Create a digraph $D = (V, E)$ as follows: $V = \{1, 2, 3, 4\}$ and $(i, j) \in E$ if $a_{ij} \geq 1$. We may assume the following axioms:

1. For all $i \neq j$, either $(i, j) \in E$ or $(j, i) \in E$. Otherwise, $A$ admits a 1-1-2 paving of norm $< 1$, violating the assumption $\alpha_3(A) = 1$.
2. For all vertices $i$, the in-degree of $i$ and the out-degree of $i$ are less than 3. Otherwise, row $i$ or column $i$ of $A$ has three entries $\geq 1 \Rightarrow \|A\| \geq 2$ (Lemma 3.16) $\Rightarrow \tilde{\alpha}_3(A) \leq \frac{2}{3} < \kappa$.

This leaves digraphs $D_{149}$, $D_{185}$, $D_{186}$, and $D_{218}$, which all have $D_{149}$ as a subgraph. Thus,

$$\|A\| \geq \left\| \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \right\| = \frac{1}{\kappa} \Rightarrow \tilde{\alpha}_3(A) \leq \kappa.$$

Now let

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then $\tilde{\alpha}_3(A) = \kappa \Rightarrow \tilde{\alpha}_3(M_{4,+}^{0}) \geq \kappa.$

□
Proposition 3.18 (6×6 nonnegative). \( \tilde{\alpha}_3(\mathbb{M}_{6,+,+}^0) \in \left[ \kappa, \frac{2}{1+\sqrt{5}} \right] \approx [0.5550, 0.6180]. \)

Proof. Suppose \( A \in \mathbb{M}_{6,+,+}^0 \), with \( \alpha_3(A) = 1 \). Create a graph \( G = (V, E) \) as follows: \( V = \{1, 2, 3, 4, 5, 6\} \) and \((i, j) \in E \) if \( a_{ij}, a_{ji} < 1 \). We may assume the following axioms:

1. \( G_61 \) is not a subgraph of \( G \). Otherwise, \( A \) has a 2-2-2 paving of norm \( < 1 \), violating the assumption \( \alpha_3(A) = 1 \).
2. By removing vertices, one cannot arrive at \( G_8 \). Otherwise, \( A \) has a 4-compression \( B \) with \( \alpha_3(B) \geq 1 \Rightarrow \|B\| \geq \frac{1}{\kappa} \Rightarrow \|A\| \geq \frac{1}{\kappa} \Rightarrow \tilde{\alpha}_3(A) \leq \kappa \).
3. \( G \) has no isolated vertices. Otherwise, if vertex \( i \) is isolated, then either row \( i \) or column \( i \) of \( A \) has at least three entries \( \geq 1 \) \( \Rightarrow \|A\| \geq \sqrt{3} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{1}{\sqrt{3}} \).
4. There does not exist a partition \( V = \{i, j, k\} \uplus \{i', j', k'\} \) such that \((r, s') \notin E, r, s \in \{i, j, k\}\). Otherwise, some \( 3 \times 3 \) submatrix of \( A \) has at least five entries \( \geq 1 \) \( \Rightarrow \|A\| \geq \frac{1}{\kappa} \Rightarrow \tilde{\alpha}_3(A) \leq \kappa \) (by exhaustive search of 0-1 \( 3 \times 3 \) matrices with five 1’s).

This leaves \( G_{114} \) and \( G_{133} \), both of which have a 5-compression of the form

\[
\begin{bmatrix}
0 & * & * & * & \\
* & 0 & * & * & \\
* & * & 0 & * & \\
* & * & * & 0 & \\
* & * & * & * & \\
\end{bmatrix},
\]

where a “*” in the \((i, j)\) position indicates that \( a_{ij} \geq 1 \) or \( a_{ji} \geq 1 \), and a “*” in the \((i, j)\) position indicates that \( a_{ij} < 1 \). Searching exhaustively over all 0-1 \( 5 \times 5 \) matrices satisfying this pattern yields \( \|A\| \geq \frac{1+\sqrt{5}}{2} \Rightarrow \tilde{\alpha}_3(A) \leq \frac{2}{1+\sqrt{5}}. \) \( \square \)
### 2,3-Pavings Summary Table

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<th>$\alpha_2(M^0_{n,sym})$</th>
<th>$\alpha_3(M^0_n)$</th>
<th>$\alpha_3(M^0_{n,s=0})$</th>
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Part 2

Supplementary Material and Tools
CHAPTER 5

Supplementary Material: 2-Pavings
CHAPTER 6

Supplementary Material: 3-Pavings

1. 4 × 4 General

Lemma 6.1. Let \( A \in M_4^0 \). If \( \alpha_3(A) = 1 \) and \( \|A\| < \sqrt{3} \), then there exists a permutation matrix \( U \in M_4 \) such that

\[
U^*AU = \begin{bmatrix}
0 & \hat{a} & b & \hat{c} \\
\hat{a} & 0 & \hat{d} & \hat{e} \\
b & \hat{d} & 0 & \hat{f} \\
\hat{c} & \hat{e} & \hat{f} & 0
\end{bmatrix},
\]

where \( |\hat{x}| \leq |\hat{x}| \) for all \( x \in \{a, b, c, d, e, f\} \). The result remains true if \( A \gg 0 \) and \( \|A\| < 2 \).

Proof. Let

\[
A = \begin{bmatrix}
0 & a_{12} & a_{13} & a_{14} \\
a_{21} & 0 & a_{23} & a_{24} \\
a_{31} & a_{32} & 0 & a_{34} \\
a_{41} & a_{42} & a_{43} & 0
\end{bmatrix}.
\]

The condition \( \alpha_3(A) = 1 \) implies that \( \max \{|a_{ij}|, |a_{ji}|\} \geq 1 \) for all \( i < j \). The condition \( \|A\| < \sqrt{3} \) (resp. \( A \gg 0 \) and \( \|A\| < 2 \)) ensures that each row and each column has at most two entries of magnitude greater than or equal to 1 (see Lemma 6.1). Conjugating by \( U_{(12)} \), if necessary, we may assume that \( |a_{12}| \geq |a_{21}| \), which we indicate as follows:

\[
A = \begin{bmatrix}
0 & \tilde{a}_{12} & \tilde{a}_{13} & \tilde{a}_{14} \\
\tilde{a}_{21} & 0 & a_{23} & a_{24} \\
\tilde{a}_{31} & a_{32} & 0 & a_{34} \\
\tilde{a}_{41} & a_{42} & a_{43} & 0
\end{bmatrix}.
\]

Case 1: Suppose \( |a_{13}| \geq |a_{31}| \). Then

\[
A = \begin{bmatrix}
0 & \tilde{a}_{12} & \tilde{a}_{13} & \tilde{a}_{14} \\
\tilde{a}_{21} & 0 & a_{23} & a_{24} \\
\tilde{a}_{31} & a_{32} & 0 & a_{34} \\
\tilde{a}_{41} & a_{42} & a_{43} & 0
\end{bmatrix}.
\]

Conjugating by \( U_{(23)} \), if necessary, we may assume that \( |a_{23}| \geq |a_{32}| \). Then

\[
A = \begin{bmatrix}
0 & \tilde{a}_{12} & \tilde{a}_{13} & \tilde{a}_{14} \\
\tilde{a}_{21} & 0 & \tilde{a}_{23} & \tilde{a}_{24} \\
\tilde{a}_{31} & \tilde{a}_{32} & 0 & \tilde{a}_{34} \\
\tilde{a}_{41} & \tilde{a}_{42} & \tilde{a}_{43} & 0
\end{bmatrix}.
\]
If $|a_{24}| \geq |a_{42}|$, then we are done. Thus, we may assume the opposite. That is,

$$A = \begin{bmatrix} 0 & \hat{a}_{12} & \hat{a}_{13} & \hat{a}_{14} \\ \hat{a}_{21} & 0 & \hat{a}_{23} & \hat{a}_{24} \\ \hat{a}_{31} & \hat{a}_{32} & 0 & \hat{a}_{34} \\ \hat{a}_{41} & \hat{a}_{42} & \hat{a}_{43} & 0 \end{bmatrix}.$$  

Conjugating by $U = U_{(1432)}$ yields

$$U^*AU = \begin{bmatrix} 0 & \hat{a}_{14} & \hat{a}_{21} & \hat{a}_{32} \\ \hat{a}_{14} & 0 & \hat{a}_{12} & \hat{a}_{13} \\ \hat{a}_{24} & \hat{a}_{21} & 0 & \hat{a}_{23} \\ \hat{a}_{34} & \hat{a}_{31} & \hat{a}_{32} & 0 \end{bmatrix}.$$  

**Case 2:** Suppose $|a_{13}| < |a_{31}|$. Then

$$A = \begin{bmatrix} 0 & \hat{a}_{12} & \hat{a}_{13} & \hat{a}_{14} \\ \hat{a}_{21} & 0 & a_{23} & a_{24} \\ \hat{a}_{31} & a_{32} & 0 & a_{34} \\ \hat{a}_{41} & a_{42} & a_{43} & 0 \end{bmatrix}.$$  

**Case 2.1:** If $|a_{14}| \geq |a_{41}|$, then

$$A = \begin{bmatrix} 0 & \hat{a}_{12} & \hat{a}_{13} & \hat{a}_{14} \\ \hat{a}_{21} & 0 & a_{23} & a_{24} \\ \hat{a}_{31} & a_{32} & 0 & a_{34} \\ \hat{a}_{41} & a_{42} & a_{43} & 0 \end{bmatrix}.$$  

Conjugating by $U_{(34)}$ yields

$$U_{(34)}^*AU_{(34)} = \begin{bmatrix} 0 & \hat{a}_{14} & \hat{a}_{12} \\ \hat{a}_{21} & 0 & a_{24} \\ \hat{a}_{31} & a_{32} & 0 \\ \hat{a}_{41} & a_{42} & a_{43} \end{bmatrix},$$  

and we may proceed as in Case 1.

**Case 2.2:** If $|a_{14}| < |a_{41}|$, then

$$A = \begin{bmatrix} 0 & \hat{a}_{12} & \hat{a}_{13} & \hat{a}_{14} \\ \hat{a}_{21} & 0 & a_{23} & a_{24} \\ \hat{a}_{31} & a_{32} & 0 & a_{34} \\ \hat{a}_{41} & a_{42} & a_{43} & 0 \end{bmatrix}.$$  

Conjugating by $U_{(34)}$ if necessary, we may assume that $|a_{34}| \geq |a_{43}|$. Then

$$A = \begin{bmatrix} 0 & \hat{a}_{12} & \hat{a}_{13} & \hat{a}_{14} \\ \hat{a}_{21} & 0 & a_{23} & a_{24} \\ \hat{a}_{31} & a_{32} & 0 & \hat{a}_{34} \\ \hat{a}_{41} & \hat{a}_{42} & \hat{a}_{43} & 0 \end{bmatrix}.$$  

**Case 2.2.1:** If $|a_{24}| \geq |a_{42}|$, then

$$A = \begin{bmatrix} 0 & \hat{a}_{12} & \hat{a}_{13} & \hat{a}_{14} \\ \hat{a}_{21} & 0 & a_{23} & a_{24} \\ \hat{a}_{31} & \hat{a}_{32} & 0 & \hat{a}_{34} \\ \hat{a}_{41} & \hat{a}_{42} & \hat{a}_{43} & 0 \end{bmatrix}.$$
Conjugating by $U = U_{(1234)}$ yields

$$U^*AU = \begin{bmatrix} 0 & \hat{a}_{23} & \hat{a}_{24} & \tilde{a}_{21} \\ \tilde{a}_{32} & 0 & \hat{a}_{34} & \hat{a}_{31} \\ \hat{a}_{42} & \hat{a}_{43} & 0 & \tilde{a}_{41} \\ \tilde{a}_{12} & \tilde{a}_{13} & \tilde{a}_{14} & 0 \end{bmatrix}.$$

Case 2.2.2: If $|a_{24}| < |a_{42}|$, then

$$A = \begin{bmatrix} 0 & \hat{a}_{12} & \hat{a}_{13} & \hat{a}_{14} \\ \hat{a}_{21} & 0 & \hat{a}_{23} & \hat{a}_{24} \\ \hat{a}_{31} & \hat{a}_{32} & 0 & \hat{a}_{34} \\ \hat{a}_{41} & \hat{a}_{42} & \hat{a}_{43} & 0 \end{bmatrix}.$$

Conjugating by $U = U_{(13)(24)}$ yields

$$U^*AU = \begin{bmatrix} 0 & \tilde{a}_{34} & \tilde{a}_{31} & \tilde{a}_{32} \\ \tilde{a}_{43} & 0 & \tilde{a}_{41} & \tilde{a}_{42} \\ \tilde{a}_{13} & \tilde{a}_{14} & 0 & \tilde{a}_{12} \\ \tilde{a}_{23} & \tilde{a}_{24} & \tilde{a}_{21} & 0 \end{bmatrix}.$$

D149: breadth-first labeling 2134

\[
\begin{bmatrix} 0 & * & * & . \\ . & 0 & * & * \\ . & . & 0 & * \\ * & * & . & 0 \end{bmatrix}
\]

$$\inf \left\{ \begin{bmatrix} 0 & 1 & 1 & . \\ . & 0 & 1 & 1 \\ . & . & 0 & 1 \\ 1 & . & . & 0 \end{bmatrix} \right\} = \begin{bmatrix} 0 & 1 & 1 & -\frac{2}{1+\sqrt{5}} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \leq \frac{1+\sqrt{5}}{2} \approx 1.6180$$

D185: breadth-first labeling 2341

\[
\begin{bmatrix} 0 & * & * & . \\ . & 0 & * & * \\ . & . & 0 & * \\ * & * & . & 0 \end{bmatrix}
\]

$$\inf \left\{ \begin{bmatrix} 0 & 1 & 1 & . \\ . & 0 & 1 & 1 \\ . & 1 & 0 & 1 \\ 1 & . & . & 0 \end{bmatrix} \right\} = \begin{bmatrix} 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \approx \sqrt{3} \approx 1.7321$$

Remark 6.2. Although this example doesn’t satisfy the hypotheses of Lemma 6.1, it satisfies the conclusion. Also, the extreme example doesn’t satisfy the graph theory, since $|.| < 1$.

D186: breadth-first labeling 3124

\[
\begin{bmatrix} 0 & * & * & . \\ . & 0 & * & * \\ * & . & 0 & * \\ * & * & . & 0 \end{bmatrix}
\]
Remark 6.3. Notice that this is a circulant. Best among circulants?
CHAPTER 7

Tools

1. Universal Selfadjoint 3-Identity and consequences

Lemma 7.1 (Universal Selfadjoint 3-Identity). Arbitrary $3 \times 3$ selfadjoint trace zero matrices $S$ satisfy:

$$\frac{|S|^2}{2||S||^2} + \frac{|\text{Det } S|}{||S||^3} = 1$$

Proof. Since all trace zero finite (or trace class) matrices have a basis in which their representation has zero diagonal, without loss of generality we can assume $S$ has the form:

$$S = \begin{pmatrix} 0 & a & b \\ \pi & 0 & c \\ b & \pi & 0 \end{pmatrix}$$

and by computation, the characteristic polynomial:

$$c_\lambda(S) = \text{det}(\lambda - S) = \lambda^3 - 2 \text{Re } \pi bc - \lambda(|a|^2 + |b|^2 + |c|^2)$$

$$= \lambda^3 - (|a|^2 + |b|^2 + |c|^2)\lambda - 2 \text{Re } a bc$$

$$= \lambda^3 - \frac{|S|^2}{2}\lambda - \text{Det } S.$$ 

An alternative way to see this is that the characteristic polynomial has the form $\lambda^3 + p\lambda^2 + q\lambda + r$, with $p = 0$ because the sum of the roots is the trace of $S$, the latter also implying

$$q = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = \frac{1}{2}((\lambda_1 + \lambda_2 + \lambda_3)^2 - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)) = -\frac{|S|^2}{2}$$

where $\lambda_j$, $j = 1, 2, 3$ denotes its roots, and $r = -\lambda_1\lambda_2\lambda_3 = -\text{det } S$.

Since $S$ is selfadjoint, $\lambda = \pm||S||$ is an eigenvalue of $S$. Also, because this is the largest eigenvalue in modulus and $S$ has trace zero, the other two real eigenvalues are opposite this in sign making their product, $\text{Det } S$, the same sign as $\lambda$. Hence $(\pm ||S||)^3 = \frac{|S|^2}{2}(\pm ||S||) + (\pm |\text{Det } S|)$, whence the Universal Selfadjoint 3-Identity in either case. □

Corollary 7.2 (Universal Selfadjoint 3-Identity consequences). For arbitrary $3 \times 3$ selfadjoint trace zero matrices $S$,

$$||S|| = 1 \iff \frac{|S|^2}{2} + |\text{Det } S| = 1.$$ 

For greater or less than 1, the respective conditions are equivalent. A necessary condition for equality is $3/2 \leq ||S||^2 \leq 2$. 

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7. TOOLS

Proof. The Universal Selfadjoint 3-Identity, \( \frac{\|S\|^2}{2\|S\|^2} + \frac{|\text{Det } S|}{3\|S\|^3} = 1 \), implies that if \( \|S\| > 1 \) then \( \frac{\|S\|^2}{2\|S\|^2} + |\text{Det } S| > 1 \), and likewise, if \( \|S\| < 1 \) then \( \frac{\|S\|^2}{2\|S\|^2} + |\text{Det } S| < 1 \). Therefore \( \|S\| = 1 \) if and only if \( \frac{\|S\|^2}{2\|S\|^2} + |\text{Det } S| = 1 \).

Moreover, if \( \frac{\|S\|^2}{2\|S\|^2} + |\text{Det } S| = 1 \), then \( \|S\| = 2 \). Also in this case when \( \|S\| = 1 \), \( \|S\| \geq \frac{2}{3} \), \( \|S\|^2 = \frac{2}{3} \) is the \( n = 3 \), \( p = 2 \) case of Proposition 7.5.

2. Universal Selfadjoint 4-Identity and consequences

Universal Selfadjoint 4-Identity (for \( 4 \times 4 \) selfadjoint zero-trace):

\[
\frac{\|S\|^2}{2\|S\|^2} + \frac{|\text{Tr } S^3|}{3\|S\|^3} - \frac{|\text{Det } S|}{4\|S\|^4} = 1
\]

Unpolished and unverified work (for proofs see file UniversalIdentities.Tex):

Consequence: Since \( \frac{|\text{Det } S|}{4\|S\|^4} \leq 1 \)

\[
\frac{\|S\|^2}{2\|S\|^2} + \frac{|\text{Tr } S^3|}{3\|S\|^3} \leq 2
\]

Separate Fact (\( \|S\|^3 \geq \frac{n}{2}\|S\|^2 \)): \( \|S\|^3 \geq \frac{4}{3}\|S\|^2 \) so \( \frac{\|S\|^2}{2\|S\|^2} \geq \frac{2}{3} \)

Implying: \( \frac{|\text{Tr } S^3|}{3\|S\|^3} \leq \frac{4}{3} \)

(Trivially also follows generally from Hölder: \( |\text{Tr } S^3|^{1/3} \leq \|S\|_3 \leq 4^{1/3}\|S\|) \)

Development of Universal Selfadjoint 4-Identity:

Let \( S \) denote a \( 4 \times 4 \) selfadjoint zero-trace matrix with eigenvalues

\[
1 = \lambda_1 \geq |\lambda_2| \geq |\lambda_3| \geq |\lambda_4|.
\]

\( c_\lambda(S) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4) \)

\[
= \lambda^4 - (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)\lambda^3 + (\sum_{i<j} \lambda_i\lambda_j)\lambda^2 - (\sum_{i<j<k} \lambda_i\lambda_j\lambda_k)\lambda + \lambda_1\lambda_2\lambda_3\lambda_4
\]

\[
= \lambda^4 + p\lambda^2 - q\lambda + r
\]

SUMMARY: NASC for \( \|S\| = 1 \) (unverified)

1. \( p \geq \frac{2}{3} \)
2. \( p + |q| + r = 1 \)
3. \( 0 \leq p + |q| \leq 2 \) (equivalent to \( \text{product of roots} \leq 1 \))
4. When \( p < 1, \frac{20}{27} - \frac{3}{3}p - \frac{2}{27}(3p - 2)^{3/2} \leq q \leq \frac{20}{27} - \frac{2}{3}p + \frac{2}{27}(3p - 2)^{3/2} \).
5. When \( p \geq 1, 0 \leq q \leq \frac{20}{27} - \frac{2}{3}p + \frac{2}{27}(3p - 2)^{3/2} \).

(4-5: \( \max(0, \frac{20}{27} - \frac{3}{3}p - \frac{2}{27}(3p - 2)^{3/2}) \leq q \leq \frac{20}{27} - \frac{2}{3}p + \frac{2}{27}(3p - 2)^{3/2} \))
3. Operator Norm/p-Norm Comparisons

**Proposition 7.3 (Operator Norm/p-Norm).** If $A$ is a finite rank selfadjoint trace 0 matrix and

$$k = \# \text{ strictly positive eigenvalues} - \# \text{ strictly negative eigenvalues},$$

then for $p \geq 1$,

$$||A||_p \leq (\text{rank } A - k)^{1/p} ||A||$$

(Sharp example: $\text{diag} (-1, 1)$)

(Sharp asymptotically: $\text{diag}(\pm 1, \ldots, \pm 1(\frac{\text{rank } A - k - 2}{2} \text{ pairs of them}), 1, -\frac{k}{k+1}, -\frac{1}{k(k+1)}$, \ldots, $-\frac{1}{k(k+1)}$; note: $\text{rank } A - k \text{ must be even}$)

**Proof.** Easy proof for $p = 2$ case:

If $<\lambda_j>$ are the (real) eigenvalues of $A$, then

$$\sum_{1}^{n} |\lambda_j|^p = \sum_{1}^{n} |\lambda_j|^{p-2} |\lambda_j|^2 \leq |\lambda_1|^{p-2} \sum_{1}^{n} |\lambda_j|^2 \leq |\lambda_1|^{p-2} (n - k) |\lambda_1|^2 = (n - k) |\lambda_1|^p.$$

For all $p \geq 1$, we describe informally the following variational approach:

Maximize $\sum |\lambda_j|^p$ subject to $\lambda_1 + \cdots + \lambda_n = 0$.

Without loss of generality, $A \neq 0$, $||A|| \leq 1$ and $tr A \neq 0$ implies that for some $n > m \geq 1$ the eigenvalues of $A$ have the $[-1, 1]$ distribution:

$$-1 \leq \lambda_n \leq \cdots \leq \lambda_{m+1} < 0 < \lambda_m \leq \cdots \leq \lambda_1 \leq 1,$$

We induct on $n - k$. Since $A \neq 0$, $n - k > 0$ and is even and so $n - k \geq 2$.

Increase $\lambda_1$ and decrease $\lambda_n$ equally so to preserve the trace, until one of them reaches 1 or $-1$, respectively. (Increasing both moduli increases the sum $\sum |\lambda_j|^p$ and so permits reduction of the proof to this case.) If they both reach 1 or $-1$, then dropping them leaves $k$ invariant and reduces to the $n - k - 2$ case.

If now $\lambda_1 = 1$ and $\lambda_n > -1$ (handle the reverse case the same), decrease $\lambda_n$ and increase $\lambda_{n-1}$ equally to preserve their sum. Elementary calculus shows that this will increase $|\lambda_n|^p + |\lambda_{n-1}|^p$. Continue this until either $\lambda_n$ reaches $-1$ or $\lambda_{n-1}$ reaches $\lambda_{n-2}$. If the former, then drop $\lambda_n$ and $\lambda_1$, and again apply the induction hypothesis. If the latter, then decrease both until $\lambda_n$ reaches $-1$ or both $\lambda_{n-1}$ and $\lambda_{n-2}$ reaches $\lambda_{n-3}$, and so on. This process will increase $\sum |\lambda_j|^p$ and unless $m = 1$, one has $m > 1$ or equivalently, $\lambda_n + \cdots + \lambda_{m+1} < -1$ implying that eventually in this process $\lambda_n$ will reach $-1$ so we can apply again the induction hypothesis while preserving $k$. If $m = 1$, then this process ends in one pair of $+1$ with sum 2 so $\sum_1^n |\lambda_j|^p \leq 2 \leq n - k$.

**Corollary 7.4.** If $A$ is an $n \times n$ selfadjoint trace 0 matrix with $n$ odd, then $||A||_2 \leq \sqrt{n - 1} ||A||$. 

Proposition 7.5. If $A$ is an $n \times n$ selfadjoint trace 0 matrix and $p \geq 1$ (or more generally rank $A = n$), then

$$||A||_p \geq [1 + \frac{1}{(n-1)^{p-1}}]^{1/p} ||A||$$

with equality iff $A = c \text{diag}(-1, \frac{1}{n-1}, \ldots, \frac{1}{n-1})$.

Proof. Suffices to show the sequence analog for $\lambda_1 + \cdots + \lambda_n = 0$, all $\lambda_j$ real. Since the inequality is obvious for $p = 1$, needing selfadjoint with trace 0 to see it, we can assume without loss of generality that $p > 1$. Then

$$|\lambda_1| = \left| - \sum_{2}^{n} \lambda_j \right| \leq ||\mathbf{1}||_p' ||\lambda||_p$$

where $\lambda := < \lambda_j >_{2 \leq j \leq n}$, $\mathbf{1} := < 1 >_{2 \leq j \leq n}$, and $\frac{1}{p} + \frac{1}{p'} = 1$, i.e., $\frac{p}{p'} = p - 1$. Equality holds if and only if $\lambda$ is a constant multiple of $\mathbf{1}$. (This is the $p$-case for Cauchy-Schwartz equality which I presume holds true for $p \neq 2$ like it does for $p = 2$—except I don’t know a reference.) So

$$|\lambda_1|^p \leq (n-1)^{p/p'} \sum_{2}^{n} |\lambda_j|^p = (n-1)^{p-1} \sum_{2}^{n} |\lambda_j|^p.$$  

Adding $(n-1)^{p-1}|\lambda_1|^p$ to both sides yields: $[1 + (n-1)^{p-1}]||A||^p \leq (n-1)^{p-1}||A||^p$, from which (iii) follows. The case for equality also follows from the previous comment about equality. □
3. OPERATOR NORM/P-NORM COMPARISONS 39

COROLLARY 7.6. If every \( k \)-compression of \( A \in \mathbb{M}_{n,sa}^0 \) has norm \( \geq 1 \), then

\[
\|A\| \geq \begin{cases} 
\frac{\sqrt{n-1}}{\sqrt{n}} & n \text{ even} \\
\frac{\sqrt{n}}{n} & n \text{ odd} 
\end{cases}
\]

**Proof.** Denote by \( \Pi_k \) the set of all \( k \)-compressions of \( A \).

Then \( \|B\|^2 \leq \frac{k-1}{k} \|B\|^2 \) for all \( B \in \Pi_k \) by Proposition 7.5 (\( p = 2 \) & take \( n \) to be \( k \)).

Then

\[
\left( \begin{array}{c} n \\ k \end{array} \right) \leq \sum_{B \in \Pi_k} \|B\|^2 \leq \frac{k-1}{k} \sum_{B \in \Pi_k} \|B\|^2_{HS} = \frac{k-1}{k} \left( \frac{n-2}{k-2} \right) \|A\|^2_{HS} \leq (n \text{ or } n-1) \frac{k-1}{k} \left( \frac{n-2}{k-2} \right) \|A\|^2.
\]

Thus,

\[
\|A\|^2 \geq \left( \begin{array}{c} n \\ k \end{array} \right) \frac{(n-1)}{k} \frac{(n-2)}{k-2} = \frac{\sqrt{n-1}}{k-1} \text{ or } \frac{\sqrt{n}}{k-1}.
\]

\( \square \)

COROLLARY 7.7. If \( \bar{\alpha}_2(\mathbb{M}_{n-k,sa}^0) < \bar{\alpha}_3(\mathbb{M}_{n,sa}^0) \) and

\[
\bar{\alpha}_3(\mathbb{M}_{n,sa}^0) \cap \{ \text{all zero-diagonals with } \pm 1 \text{ off diagonal entries} \} \left( \begin{array}{c} k-1 \\ \sqrt{n-1} \end{array} \right), \quad n \text{ even} \\
\left( \begin{array}{c} k-1 \\ \sqrt{n} \end{array} \right), \quad n \text{ odd}
\]

then

\[
\bar{\alpha}_3(\mathbb{M}_{n,sa}^0) \left( \begin{array}{c} k-1 \\ \sqrt{n-1} \end{array} \right), \quad n \text{ even} \\
\left( \begin{array}{c} k-1 \\ \sqrt{n} \end{array} \right), \quad n \text{ odd}.
\]

**Proof.** Fix an extremal \( A = A_n \), that is, \( \bar{\alpha}_3(\mathbb{M}_{n,sa}^0) = \frac{\alpha_3(A)}{\|A\|} \) and without loss of generality assume \( \alpha_3(A) = 1 \) and \( \|A\| = \frac{1}{\bar{\alpha}_3(\mathbb{M}_{n,sa}^0)} \).

Either \( \|B\| < 1 \) for some \( k \)-compression or every \( k \)-compression \( B \) of \( A \) has norm \( \geq 1 \).

Assume first \( \|B\| < 1 \) for some \( k \)-compression \( B = PAQ \). Because \( \alpha_3(A) = 1 \), every 3-paving has norm \( \geq 1 \) and by definition, \( \bar{\alpha}_2(\mathbb{M}_{n-k,sa}^0) \geq \frac{\alpha_2((I-P)A(I-P))}{\|A\|} \) so

\[
\|(I-P)A(I-P)\| \geq \frac{\alpha_2((I-P)A(I-P))}{\bar{\alpha}_2(\mathbb{M}_{n-k,sa}^0)}.
\]

So if additionally \( \|B\| < 1 \) and \( \alpha_3(A) = 1 \), then \( \alpha_2((I-P)A(I-P)) = 1 \) so all 2-pavings of \( (I-P)A(I-P) \) have norm \( \geq 1 \), in which case

\[
\|A\| \geq \|(I-P)A(I-P)\| \geq \frac{1}{\bar{\alpha}_2(\mathbb{M}_{n-k,sa}^0)} > \frac{1}{\bar{\alpha}_3(\mathbb{M}_{n,sa}^0)}.
\]

The last \( > \) by hypothesis, contradicting \( \bar{\alpha}_3(\mathbb{M}_{n,sa}^0) = \frac{\alpha_3(A)}{\|A\|} = \frac{1}{\bar{\alpha}_3(\mathbb{M}_{n,sa}^0)} \).

On the other hand, if every \( k \)-compression \( B \) of \( A \) has norm \( \geq 1 \), then the displayed inequality in Corollary 7.6 becomes equality throughout:

\[
\left( \begin{array}{c} n \\ k \end{array} \right) \leq \sum_{B \in \Pi_k} \|B\|^2 \leq \sum_{B \in \Pi_k} \|B\|^2_{HS} = \frac{k-1}{k} \left( \frac{n-2}{k-2} \right) \|A\|^2_{HS} = (n \text{ or } n-1) \frac{k-1}{k} \left( \frac{n-2}{k-2} \right) \|A\|^2.
\]

So each \( \|B\|^2 = \frac{k-1}{k} \|B\|^2_{HS} \). Now apply Lemma 3.12 so that

\[
A \equiv S \in \mathbb{M}_{n,sa}^0 \cap \{ \text{all zero-diagonals with } \pm 1 \text{ off diagonal entries} \}
\]

and apply the hypothesis to \( S \) to contradict the extremality of \( A \). \( \square \)
4. Operator Norm/Hilbert-Schmidt Norm Comparisons

**Lemma 7.8.** Let \( A \in \mathbb{M}_n \). Then
\[
\| A \| \leq \| A \|_{HS} \leq \sqrt{n} \| A \|.
\]

Furthermore,
1. \( \| A \| = \| A \|_{HS} \) if and only if \( \text{rank}(A) \leq 1 \).
2. \( \| A \|_{HS} = \sqrt{n} \| A \| \) if and only if \( A \) is a scalar multiple of a unitary.

**Proof.** The inequalities are well-known and easy to prove. Now let
\[
A = U\Sigma V^*
\]
be a singular value decomposition of \( A \) (i.e. \( U, V \) are unitary and \( \Sigma = \text{diag}(\sigma_1, \sigma_2, ..., \sigma_n) \), where \( \sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n \geq 0 \)). Assume \( \| A \| = \| A \|_{HS} \). Then
\[
\sigma_1^2 = \| A \|^2 = \| A \|_{HS}^2 = \sum_{i=1}^{n} \sigma_i^2 \Rightarrow \sigma_2 = \sigma_3 = ... = \sigma_n = 0.
\]
Thus, \( A = \sigma_1 u_1 v_1^* \), where \( u_1 \) and \( v_1 \) are the first columns of \( U \) and \( V \), respectively.

Hence, \( \text{rank}(A) \leq 1 \). Conversely, if \( \text{rank}(A) \leq 1 \), then
\[
\sigma_2 = \sigma_3 = ... = \sigma_n = 0 \Rightarrow \| A \| = \| A \|_{HS}.
\]

Now assume \( \| A \|_{HS} = \sqrt{n} \| A \| \). Then
\[
\sum_{i=1}^{n} \sigma_i^2 = \| A \|^2_{HS} = n \| A \|^2 = n \sigma_1^2 \Rightarrow \sigma_1 = \sigma_2 = ... = \sigma_n.
\]
Thus, \( A = \sigma_1 U V^* \), which is a scalar multiple of a unitary. Conversely, if \( A = \alpha W \), where \( \alpha \in \mathbb{C} \) and \( W \) is a unitary, then
\[
\| A \|^2_{HS} = \text{Tr}(A^* A) = |\alpha|^2 \text{Tr}(W^* W) = |\alpha|^2 \text{Tr}(I) = n|\alpha|^2 = n \| A \|^2.
\]

**Corollary 7.9.** If every 3-compression of \( A \in \mathbb{M}_7^0 \) has norm \( \geq 1 \), then
\[
\| A \| \geq \sqrt{n - 1} \frac{1}{k(k - 1)}.
\]
Equality occurs if and only if \( A \) is a multiple of a unitary and every \( k \)-compression of \( A \) has rank one.

**Proof.** Denote by \( \Pi_k \) the set of all \( k \)-compressions of \( A \). Then
\[
\binom{n}{k} \leq \sum_{B \in \Pi_k} \| B \|^2 \leq \sum_{B \in \Pi_k} \| B \|_{HS}^2 = \binom{n - 2}{k - 2} \| A \|_{HS}^2 \leq n \binom{n - 2}{k - 2} \| A \|^2.
\]
Thus,
\[
\| A \|^2 \geq \frac{\binom{n}{k}}{\binom{n - 2}{k - 2}} = \frac{n - 1}{k(k - 1)}.
\]
The stated equality condition follows immediately from Lemma 7.8.

**Corollary 7.10.** If every 3-compression of \( A \in \mathbb{M}_7^0 \) has norm \( \geq 1 \), then \( \| A \| > 1 \).
Proof. By Lemma 7.9, \[ \|A\|_2^2 \geq \frac{7 - 1}{3(3 - 1)} = 1. \]

Suppose \( \|A\| = 1 \). Again by Lemma 7.9, \( A \) is unitary and every 3-compression of \( A \) has rank one. It follows that every 3-compression of \( A \) has exactly two zero columns or exactly two zero rows. Consider \( A_{123} \), the \( \{1, 2, 3\} \)-compression of \( A \).

Without loss of generality, we may assume that the second and third columns of \( A_{123} \) are zero. It follows that the first column of \( A_{123} \) has norm 1. Thus,

\[
A = \begin{bmatrix}
0 & 0 & 0 & * & * & * \\
0 & 0 & * & * & * & *\\
0 & * & * & 0 & * & * \\
0 & * & * & * & 0 & * \\
0 & * & * & * & 0 & *
\end{bmatrix},
\]

where \( |a_{21}|^2 + |a_{31}|^2 = 1 \). Conjugating by \( U_{(23)} \), if necessary, we may assume that \( a_{21} \neq 0 \). Case 1: Suppose \( |a_{21}| = 1 \). By considering, in order, \( A_{123} \), \( A_{124} \), \( A_{125} \), \( A_{126} \), and \( A_{127} \), we have that

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & * & * & * & * \\
0 & 0 & * & 0 & * & * \\
0 & 0 & * & * & 0 & * \\
0 & 0 & * & * & 0 & 0
\end{bmatrix}.
\]

Considering \( A_{234} \), we have that either \( |a_{34}| = 1 \) or \( |a_{43}| = 1 \). Conjugating by \( U_{(34)} \), if necessary, we may assume the former. Considering, in order, \( A_{234} \), \( A_{345} \), \( A_{346} \), and \( A_{347} \), we have that

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & * & 0 \\
0 & 0 & 0 & 0 & * & 0
\end{bmatrix}.
\]

But then \( \|A_{235}\| = 0 \), a contradiction.

Case 2: Suppose \( |a_{21}| < 1 \). By considering, in order, \( A_{124} \), \( A_{234} \), and \( A_{345} \), we have that

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & * & * \\
0 & 0 & a_{24} & 0 & 0 & 0 \\
0 & 0 & a_{34} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0 & *
\end{bmatrix}.
\]
where \(|a_{21}|^2 + |a_{24}|^2 = 1\) and \(|a_{24}|^2 + |a_{44}|^2 = 1\). But then \(\|A_{345}\| < 1\), a contradiction.

Lemma 7.11. Let \(A \in M_4^0\). If every 2-2 paving of \(A\) has norm \(\geq 1\), then either \(\|A\| > 1\) or, up to permutation,

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & a & 0 \\
0 & 0 & 0 & b \\
0 & c & 0 & 0
\end{bmatrix},
\]

where \(|a| = |b| = |c| = 1\).

Proof. Assume \(\|A\| = 1\). Create a graph \(G = (V, E)\) as follows: \(V = \{1, 2, 3, 4\}\) and \((i, j) \in E\) if \(|a_{ij}|, |a_{ji}| < 1\). We may assume the following axioms:

1. \(G_{11}\) is not a subgraph of \(G\). Otherwise, \(A\) has a 2-2 paving of norm \(< 1\).
2. For all \(i, \deg(i) > 0\). Otherwise, either row \(i\) or column \(i\) of \(A\) has at least two entries of modulus \(\geq 1 \Rightarrow \|A\| \geq \sqrt{2}\).

This leaves \(G_{13}\), which proves the result. \(\square\)
5. Averaging and Constrained Averaging

Let $A^* = A = (a_{ij})$, $E(A) = 0$, with the reduction assumption for $M_{7,s,a}^0$ that the $B$'s range over all the $3 \times 3$ zero-diagonal matrices with norm at least 1 (in which case each Hilbert-Schmidt norm is at least $\frac{2}{3}$) or in the case of constrained averaging, all the $B$'s with diagonal projection not containing prescribed $i, j$ pairs.

The following weighted formulas for the Hilbert-Schmidt norm of a $7 \times 7$ zero-diagonal selfadjoint matrix in terms of the Hilbert-Schmidt norms of some or all of its 3-diagonal compressions $PAP$ for averaging and constrained averaging are obtained by careful groupings of triplet integer subsets of $[1, 7]$ to compensate for overcounting due to multiple occurrences, analogous to the elementary counting formula for finite sets: $|A \cup B| = |A| + |B| - |A \cap B|$. (0)

\[
6||A||^2 \geq ||A||^2_{HS} = \frac{1}{5} \sum_{all} \frac{35}{2} ||B||^2_{HS} \quad \text{(Averaging)}
\]

(12)

\[
6||A||^2 \geq ||A||^2_{HS} = 2|a_{12}|^2 + \left( \frac{1}{4} \sum_{134-267} + \frac{1}{6} \sum_{345-567} \right) ||B||^2_{HS} \quad \text{(Constrained Averaging here and below)}
\]

(row)

\[
6||A||^2 \geq ||A||^2_{HS} = 2||Ac_1||^2 + \frac{1}{4} \sum_{1 \notin B} \frac{20}{4} ||B||^2_{HS}
\]

(12,23)

\[
6||A||^2 \geq ||A||^2_{HS} = 2|a_{12}|^2 + 2|a_{23}|^2 + \left( \frac{1}{4} \sum_{1 \in B, 2 \notin B} + \frac{1}{3} \sum_{1 \notin B, 2 \notin B} + \frac{1}{6} \sum_{1 \notin B, 3 \notin B} + \frac{1}{12} \sum_{1,2 \notin B, 3 \notin B} \right) ||B||^2_{HS}
\]

(12,13)

\[
6||A||^2 \geq ||A||^2_{HS} = 2|a_{12}|^2 + 2|a_{13}|^2 + \left( \frac{1}{3} \sum_{1 \in B, 2 \notin B} + \frac{1}{4} \sum_{1 \notin B, 2 \notin B} + \frac{1}{6} \sum_{1 \notin B, 2 \notin B} \right) ||B||^2_{HS}
\]

(12,23,34)

\[
6||A||^2 \geq ||A||^2_{HS} = 2|a_{12}|^2 + 2|a_{23}|^2 + 2|a_{34}|^2 + \left( \frac{1}{3} \sum_{135-147, all B \notin B'} + \frac{1}{6} \sum_{156-167, 456-467} + \frac{1}{6} \sum_{567} \right) ||B||^2_{HS}
\]
Application of constrained averaging:

If $|a_{ij}| \geq 1$ (wlog $i, j = 1, 2$) and $A$ satisfies the 3-compression reduction given above, then by (12),

$$6\|A\|^2 \geq \|A\|_{HS}^2 = 2|a_{12}|^2 + \left( \frac{1}{4} \sum_{134-267}^{20} + \frac{1}{6} \sum_{345-567}^{10} \right) \|B\|_{HS}^2$$

$$\geq 2 + \left( \frac{1}{4} \sum_{134-267}^{20} + \frac{1}{6} \sum_{345-567}^{10} \right) \frac{3}{2} \|B\|$$

$$\geq 2 + \left( \frac{20}{4} + \frac{10}{6} \right) \frac{3}{2} = 2 + \left( \frac{5 + 5}{3} \right) \frac{3}{2} = 12$$

So $6\|A\|^2 \geq 12$, $\|A\| \geq \sqrt{2}$, $\tilde{\alpha}_3(A) \leq \frac{1}{\sqrt{2}} \approx .7071$, smaller than the $\tilde{\alpha}_3(M_7^{0, sa})$-table upper range in $[\frac{\sqrt{8}}{3}, \frac{\sqrt{7}}{3}] = [.6667, .7559)$. This then rules out entries with larger than 1 modulus for an extremal bad paver in case one succeeds in proving $\tilde{\alpha}_3(M_7^{0, sa}) \in \left( \frac{\sqrt{8}}{3}, \frac{\sqrt{7}}{3} \right)$.

Moreover, since $\frac{1}{\|A\|} = \tilde{\alpha}_3(M_7^{0, sa})$, if $A$ were extremal, and wlog $|a_{12}| = \max_{i,j} |a_{ij}|$, then $\|A\|^2 = \frac{1}{\tilde{\alpha}_3(M_7^{0, sa})^2} \in \left( \frac{7}{4}, \frac{9}{4} \right)$ and

$$\|A\|^2 \geq \frac{1}{6} \|A\|_{HS}^2 = \frac{1}{3} |a_{12}|^2 + \frac{1}{6} \left( \frac{1}{4} \sum_{134-267}^{20} + \frac{1}{6} \sum_{345-567}^{10} \right) \|B\|_{HS}^2$$

$$\geq \frac{|a_{12}|^2}{3} + \left( \frac{1}{4} \sum_{134-267}^{20} + \frac{1}{6} \sum_{345-567}^{10} \right) \frac{3}{2} \|B\|$$

$$\geq \frac{|a_{12}|^2}{3} + \frac{1}{6} \left( \frac{20}{4} + \frac{10}{6} \right) \frac{3}{2} = \frac{|a_{12}|^2}{3} + \frac{5}{3} > \frac{9}{4}$$

leads to the contradiction: $\tilde{\alpha}_3(M_7^{0, sa}) = \frac{1}{\|A\|} < \frac{2}{3}$. Hence

$$|a_{12}|^2 \leq \frac{27}{4} - 5 = \frac{7}{4}, \text{ i.e., } \max_{i,j} |a_{ij}| \leq \frac{\sqrt{7}}{2} < \|A\|.$$
Bibliography