EQUIVALENTS OF THE KADISON-SINGER PROBLEM

PETER G. CASAZZA AND DAN EDIDIN

Abstract. In a series of papers it was recently shown that the 1959 Kadison-Singer Problem in C^*-Algebras is equivalent to fundamental unsolved problems in a dozen areas of research in pure mathematics, applied mathematics and engineering. Because of the length and depth of these papers, it has been difficult for people in the various impacted areas to get an overview of how their problems relate to the other areas of research. In this note we will give a short introduction to the various equivalences of the Kadison-Singer Problem for those inside and outside these fields who would like to appreciate these recent advances. We will also introduce several new equivalences of the Kadison-Singer Problem including, for the first time, connections to Algebraic Geometry.

1. The Kadison-Singer Problem in C^*-Algebras

The 1959 Kadison-Singer Problem [56] has proved to be one of the most intractible problems in mathematics. To state the problem, we need some notation. A **state** of a von Neumann algebra $\mathcal{R}$ is a linear functional f on $\mathcal{R}$ for which $f(I) = 1$ and $f(T) \geq 0$ whenever $T \geq 0$ (whenever T is a positive operator). The set of states of $\mathcal{R}$ is a convex subset of the dual space of $\mathcal{R}$ which is compact in the ω^*-topology. By the Krein-Milman theorem, this convex set is the closed convex hull of its extreme points. The extremal elements in the space of states are called the **pure states** (of $\mathcal{R}$).

Kadison-Singer Problem (KS)**.** *Does every pure state on the (abelian) von Neumann algebra $\mathbb{D}$ of bounded diagonal operators on ℓ_2 have a unique extension to a (pure) state on $B(\ell_2)$, the von Neumann algebra of all bounded linear operators on the Hilbert space ℓ_2?*

This problem grew out of a mistake Dirac made in his Quantum Mechanics book [31]. Dirac wanted to find a "representation" (an orthonormal basis) for a compatible family of observables (a commutative family of self-adjoint operators). On pages 74–75 of [31] Dirac states:

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> "To introduce a representation in practice
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> (i) We look for observables which we would like to have diagonal either because we are interested in their probabilities or for reasons of mathematical simplicity;
> (ii) We must see that they all commute — a necessary condition since diagonal matrices always commute;
> (iii) We then see that they form a complete commuting set, and if not we add some more commuting observables to make them into a complete commuting set;
> (iv) We set up an orthogonal representation with this commuting set diagonal.
>
> **The representation is then completely determined ... by the observables that are diagonal ...**"

We added the emphasis. In the case of $\mathbb{D}$, the representation is $\{e_i\}_{i\in I}$, the orthonormal basis of l_2. But what happens if our observables have "ranges" (intervals) in their spectra? This led Dirac to introduce his famous δ-function — vectors of "infinite length." From a mathematical point of view, this is problematic. What we need is to replace the vectors e_i by some mathematical object that is essentially the same as the vector, when there is one, but gives us something precise and usable when there is only a δ-function. This leads to the "pure states" of $B(\ell_2)$ and, in particular, the (vector) pure states ω_x, given by $\omega_x(T) = \langle Tx, x\rangle$, where x is a unit vector in $\mathbb{H}$. Then, $\omega_x(T)$ is the expectation value of T in the state corresponding to x. This expectation is the average of values measured in the laboratory for the "observable" T with the system in the state corresponding to x. The pure state ω_{e_i} can be shown to be completely determined by its values on $\mathbb{D}$; that is, each ω_{e_i} has a *unique* extension to $B(\ell_2)$. But there are many other pure states of $\mathbb{D}$. (The family of all pure states of $\mathbb{D}$ with the w^*-topology is $\beta(\mathbb{Z})$, the β-compactification of the integers.) Do these other pure states have unique extensions? This is the Kadison-Singer problem (KS).

Notation for statements of problems: Problem A (or Conjecture A) **implies** Problem B (or Conjecture B) means that a positive solution to the former implies a positive solution to the latter. They are **equivalent** if they imply each other.

Notation for Hilbert spaces: Throughout, $\ell_2(I)$ will denote a finite or infinite dimensional complex Hilbert space with a fixed orthonormal basis $\{e_i\}_{i\in I}$. If I is infinite we let $\ell_2 = \ell_2(I)$, and if $|I| = n$ write $\ell_2(I) = \ell_2^n$ with fixed orthonormal basis $\{e_i\}_{i=1}^n$. For any Hilbert space $\mathbb{H}$ we let $B(\mathbb{H})$ denote the family of bounded linear operators on $\mathbb{H}$. An n-dimensional subspace of $\ell_2(I)$

will be denoted $\mathbb{H}_N$. For an operator T on any one of our Hilbert spaces, its matrix representation with respect to our fixed orthonormal basis is the collection $(\langle Te_i, e_j\rangle)_{i,j\in I}$. If $J \subset I$, the **diagonal projection** Q_J is the matrix whose entries are all zero except for the (i,i) entries for $i \in J$ which are all one. For a matrix $A = (a_{ij})_{i,j\in I}$ let $\delta(A) = \max_{i\in I} |a_{ii}|$.

There are other equivalent formulations and implications in C^*-Algebras of KS [1, 3]. One equivalent form of the Kadison-Singer Problem which has received a great deal of attention is the **relative Dixmier Property** [1, 3, 4, 5, 6, 7, 32, 11, 12, 51, 50, 52].

Relative Dixmier Property. *For every $x \in B(\mathbb{H})$ the set*

$$K(x) = \overline{co}\{wx^*w : w \in \mathbb{D}, w \text{ unitary}\}$$

has nonempty intersection with $\mathbb{D}$, in which case $K(x) \cap \mathbb{D} = \{E(x)\}$, where $E(x)$ denotes the diagonal of x (relative Dixmier property for the embedding of $\mathbb{D}$ into $B(\mathbb{H})$).

2. The Kadison-Singer Problem in Operator Theory

A significant advance on KS was made by Anderson [3] in 1979 when he reformulated KS into what is now known as the **Paving Conjecture** (See also [4, 6]). Lemma 5 of [56] shows a connection between KS and Paving.

Paving Conjecture (PC)**.** *For $\epsilon > 0$, there is a natural number r so that for every natural number n and every linear operator T on l_2^n whose matrix has zero diagonal, we can find a partition (i.e. a paving) $\{A_j\}_{j=1}^r$ of $\{1, \cdots, n\}$, such that*

$$\|Q_{A_j} T Q_{A_j}\| \le \epsilon \|T\| \quad \text{for all } j = 1, 2, \cdots, r.$$

It is important that r not depend on n in PC. We will say that an arbitrary operator T satisfies PC if $T - D(T)$ satisfies PC where $D(T)$ is the diagonal of T.

It is known [11] that the class of operators satisfying PC (the **pavable operators**) forms a closed subspace of $B(\ell_2)$. The only large classes of operators which have been shown to be pavable are “diagonally dominant” matrices [9, 10, 11, 47], matrices with all entries real and positive [51] and Toeplitz operators over Riemann integrable functions (See also [52] and Section 6). Also, in [12] there is an analysis of the paving problem for certain Schatten C_p-norms. To verify PC it suffices to verify it for any one of your favorite classes of operators [24]: Unitary operators, positive operators, orthogonal projections, self-adjoint operators, invertible operators or Gram matrices $(\langle f_i, f_j\rangle)_{i,j\in I}$ where $T : \ell_2(I) \to \ell_2(I)$ is a bounded linear operator, and $Te_i = f_i$, $\|Te_i\| = 1$ for all $i \in I$.

Akemann and Anderson [1] showed that the following conjecture implies KS.

Conjecture 2.1. *There exists $0 < \epsilon, \delta < 1$ with the following property: for any orthogonal projection P on ℓ_2^n with $\delta(P) \le \delta$, there is a diagonal projection Q such that $\|QPQ\| \le 1-\epsilon$ and $\|(I-Q)P(I-Q)\| \le 1-\epsilon$.*

It is important that ϵ, δ are independent of n in Conjecture 2.1. It is unknown if KS implies Conjecture 2.1. Anderson and Akemann [1] formulated two other conjectures which they showed implied KS. Recall that a **diagonal symmetry** is a diagonal operator with ± 1 on the diagonal.

Conjecture 2.2. *For any projection P on l_2^n there exists a diagonal symmetry S so that $\|PSP\| \le 2\delta(P)$.*

In a deep paper, Weaver [61] gave a counter-example to Conjecture 2.2. In [23] there is a simpler counter-example which relies on equiangular frames.

Conjecture 2.3. *There exists $\epsilon, \delta > 0$ such that for any projection P on l_2^n with $\delta(P) \le \delta$ there exists a diagonal symmetry S such that $\|PSP\| < 1-\epsilon$.*

Anderson and Akemann [1] showed that Conjecture 2.3 implies KS. The converse remains open at this time.

Recently, Weaver [61] provided important insight into KS by showing that a slight weakening of Conjecture 2.1 will produce a conjecture equivalent to KS. We will not give this conjecture here but instead state a recent strengthening of this result due to Casazza, Edidin, Kalra and Paulsen [23]. In [23] it is shown that KS is equivalent to this conjecture.

Conjecture 2.4. *There is a natural number r and an $\epsilon > 0$ so that for all orthogonal projections P on ℓ_2^{2n} with $1/2'$s on the diagonal, there is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \cdots, 2n\}$ so that $\|Q_{A_j} P Q_{A_j}\| \le 1-\epsilon$, for all $j = 1, 2, \cdots, r$.*

Note that Conjecture 2.4 requires us to check only projections of ℓ_2^{2n} onto n-dimensional subspaces.

3. Kadison-Singer in inner product theory

Recall that a family of vectors $\{f_i\}_{i\in I}$ is a **Riesz basic sequence** in a Hilbert space $\mathbb{H}$ if there are constants $A, B > 0$ so that for all scalars $\{a_i\}_{i\in I}$ we have:

$$A\sum_{i\in I}|a_i|^2 \le \|\sum_{i\in I} a_i f_i\|^2 \le B\sum_{i\in I}|a_i|^2.$$

We call $\sqrt{A}, \sqrt{B}$ the **lower and upper Riesz basis bounds** for $\{f_i\}_{i\in I}$. If the Riesz basic sequence $\{f_i\}_{i\in I}$ spans $\mathbb{H}$ we call it a **Riesz basis** for $\mathbb{H}$. So $\{f_i\}_{i\in I}$ is a Riesz basis for $\mathbb{H}$ means there is an orthonormal basis $\{e_i\}_{i\in I}$

so that the operator $T(e_i) = f_i$ is invertible. In particular, each Riesz basis is **bounded**. That is, $0 < \inf_{i\in I} \|f_i\| \le \sup_{i\in I} \|f_i\| < \infty$. If $\epsilon > 0$ and $A = 1-\epsilon, B = 1+\epsilon$ we call $\{f_i\}_{i\in I}$ an ϵ-**Riesz basic sequence**. If $\|f_i\| = 1$ for all $i \in I$ this is a **unit norm** Riesz basic sequence. A natural question is whether we can improve the Riesz basis bounds for a unit norm Riesz basic sequence by partitioning the sequence into subsets. Formally:

Conjecture 3.1 (R_ϵ-Conjecture). *For every $\epsilon > 0$, every unit norm Riesz basic sequence is a finite union of ϵ-Riesz basic sequences.*

The R_ϵ-Conjecture was first stated Casazza and Vershynin [29] where it was shown that KS implies this conjecture. It was recently shown in [27] that KS is equivalent to the R_ϵ-Conjecture.

Note that PC is independent of switching to equivalent norms on the Hilbert space. However, it can be shown [27] that the R_ϵ-conjecture fails for some equivalent norms.

4. Kadison-Singer in Frame Theory

Hilbert space frames were introduced by Duffin and Schaeffer [36] to address some very deep problems in nonharmonic Fourier series (see [62]). For an introduction to frame theory we refer the reader to Christensen [30].

A family $\{f_i\}_{i\in I}$ of elements of a (finite or infinite dimensional) Hilbert space $\mathbb{H}$ is called a **frame** for $\mathbb{H}$ if there are constants $0 < A \le B < \infty$ (called the **lower and upper frame bounds**, respectively) so that for all $f \in \mathbb{H}$

$$A\|f\|^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B\|f\|^2. \tag{4.1}$$

If we only have the right hand inequality in Equation 4.1 we call $\{f_i\}_{i\in I}$ a **Bessel sequence with Bessel bound B**. If $A = B$, we call this an A-**tight frame** and if $A = B = 1$, it is called a **Parseval frame**. If all the frame elements have the same norm, this is an **equal norm** frame and if the frame elements are of unit norm, it is a **unit norm frame**. It is immediate that $\|f_i\|^2 \le B$. If also $\inf \|f_i\| > 0$, $\{f_i\}_{i\in I}$ is a **bounded frame**. The numbers $\{\langle f, f_i\rangle\}_{i\in I}$ are the **frame coefficients** of the vector $f \in \mathbb{H}$. If $\{f_i\}_{i\in I}$ is a Bessel sequence, the **synthesis operator** for $\{f_i\}_{i\in I}$ is the bounded linear operator $T : \ell_2(I) \to \mathbb{H}$ given by $T(e_i) = f_i$ for all $i \in I$. The **analysis operator** for $\{f_i\}_{i\in I}$ is T^* and satisfies: $T^*(f) = \sum_{i\in I} \langle f, f_i\rangle e_i$. In particular,

$$\|T^* f\|^2 = \sum_{i\in I} |\langle f, f_i\rangle|^2, \quad \text{for all } f \in \mathbb{H},$$

and hence the smallest Bessel bound for $\{f_i\}_{i\in I}$ equals $\|T^*\|^2$. It follows that a Bessel sequence is a Riesz basic sequence if and only if T^* is onto.

The **frame operator** for the frame is the positive, self-adjoint invertible operator $S = TT^* : \mathbb{H} \to \mathbb{H}$. That is,

$$Sf = TT^*f = T\left(\sum_{i\in I}\langle f, f_i\rangle e_i\right) = \sum_{i\in I}\langle f, f_i\rangle Te_i = \sum_{i\in I}\langle f, f_i\rangle f_i.$$

In particular,

$$\langle Sf, f\rangle = \sum_{i\in I}|\langle f, f_i\rangle|^2.$$

It follows that $\{f_i\}_{i\in I}$ is a frame with frame bounds A, B if and only if $A\cdot I \le S \le B\cdot I$. So $\{f_i\}_{i\in I}$ is a Parseval frame if and only if $S = I$. **Reconstruction** of vectors in $\mathbb{H}$ is achieved via the formula:

$$\begin{aligned} f &= SS^{-1}f = \sum_{i\in I}\langle S^{-1}f, f_i\rangle f_i = \sum_{i\in I}\langle f, S^{-1}f_i\rangle f_i \\ &= \sum_{i\in I}\langle f, f_i\rangle S^{-1}f_i = \sum_{i\in I}\langle f, S^{-1/2}f_i\rangle S^{-1/2}f_i. \end{aligned}$$

It follows that $\{S^{-1/2}f_i\}_{i\in I}$ is a Parseval frame *equivalent* to $\{f_i\}_{i\in I}$. Two sequences $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ in a Hilbert space are *equivalent* if there is an invertible operator T between their spans with $Tf_i = g_i$ for all $i \in I$.

A fundamental result in frame theory was proved independently by Naimark and Han/Larson [30, 53].

Theorem 4.1. *A family $\{f_i\}_{i\in I}$ is a Parseval frame for a Hilbert space $\mathbb{H}$ if and only if there is a containing Hilbert space $\mathbb{H} \subset \ell_2(I)$ with an orthonormal basis $\{e_i\}_{i\in I}$ so that the orthogonal projection P of $\ell_2(I)$ onto $\mathbb{H}$ satisfies $P(e_i) = f_i$ for all $i \in I$.*

In his work on time-frequency analysis, Feichtinger (See [21] where this was first formally stated in the literature) noted that all of the Gabor frames he was using (see Section 7) had the property that they could be divided into a finite number of subsets which were Riesz basic sequences. This led to the conjecture:

Feichtinger Conjecture (FC)**.** *Every bounded frame (or equivalently, every unit norm frame) is a finite union of Riesz basic sequences.*

There is a significant body of work on this conjecture [9, 10, 29, 47]. Yet, it remains open even for Gabor frames. In [21] it was shown that FC is equivalent to the weak BT, and hence is implied by KS (See Section 5). In [27] it was shown that FC is equivalent to KS. In fact, we now know that KS is equivalent to the *weak* Feichtinger Conjecture: Every unit norm Bessel sequence is a finite union of Riesz basic sequences (See Section 5).

Weaver [61] also gave the following conjecture and showed that it is equivalent to PC.

Conjecture 4.2 ($\mathrm{KS_r}$). *There is a natural number r and universal constants B and $\epsilon > 0$ so that the following holds. Let $\{f_i\}_{i=1}^M$ be elements of ℓ_2^n with $\|f_i\| \le 1$ for $i = 1, 2, \cdots, M$ and suppose for every $f \in \ell_2^n$,*

$$\sum_{i=1}^{M} |\langle f, f_i\rangle|^2 = B\|f\|^2. \tag{4.2}$$

Then, there is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \cdots, n\}$ so that for all $f \in \ell_2^n$ and all $j = 1, 2, \cdots, r$,

$$\sum_{i\in A_j} |\langle f, f_i\rangle|^2 \le (B - \epsilon)\|f\|^2.$$

5. Kadison-Singer in Banach Banach Space Theory

In 1987, Bourgain and Tzafriri [14] proved a fundamental result in Banach space theory known as the **restricted invertibility principle**.

Theorem 5.1 (Bourgain-Tzafriri). *There are universal constants $A, c > 0$ so that whenever $T : \ell_2^n \to \ell_2^n$ is a linear operator for which $\|Te_i\| = 1$, for $1 \le i \le n$, then there exists a subset $\sigma \subset \{1, 2, \cdots, n\}$ of cardinality $|\sigma| \ge cn/\|T\|^2$ so that for all $j = 1, 2, \cdots, n$ and for all choices of scalars $\{a_j\}_{j\in\sigma}$,*

$$\|\sum_{j\in\sigma} a_j Te_j\|^2 \ge A \sum_{j\in\sigma} |a_j|^2.$$

Theoerem 5.1 gave rise to two conjectures in the area known as the strong and weak Bourgain-Tzafriri Conjectures. These conjectures have received a great deal of attention [15, 27, 29]. Both of these conjectures are now known to be equivalent to the Kadison-Singer Problem [27].

Bourgain-Tzafriri Conjecture (BT). *There is a universal constant $A > 0$ so that for every $B > 1$ there is a natural number $r = r(B)$ satisfying: For any natural number n, if $T : \ell_2^n \to \ell_2^n$ is a linear operator with $\|T\| \le B$ and $\|Te_i\| = 1$ for all $i = 1, 2, \cdots, n$, then there is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \cdots, n\}$ so that for all $j = 1, 2, \cdots, r$ and all choices of scalars $\{a_i\}_{i\in A_j}$ we have:*

$$\|\sum_{i\in A_j} a_i Te_i\|^2 \ge A \sum_{i\in A_j} |a_i|^2.$$

It had been "folklore" for years that KS and BT must be equivalent. But no one was quite able to actually give a proof of this fact. Recently, Casazza and Vershynin [29] gave a formal proof of the equivalence of KS and BT. Sometimes BT is called **strong BT** since there is a weakening of it called **weak BT**. In weak BT we allow A to depend upon the norm of the operator T.

Now we will present a significant strengthening of weak-BT.

Conjecture 5.2. *There is a universal constant $A>0$ and a natural number $r\in\mathbb{N}$ so that whenever P is a projection on ℓ_2^{2n} with $\|Pe_i\|^2=1/2$ for all $i=1,2,\cdots,n$ there is a partition $\{A_j\}_{j=1}^r$ of $\{1,2,\cdots,2n\}$ satisfying for all $j=1,2,\cdots,r$ and all $\{a_i\}_{i\in A_j}$*

$$\|\sum_{i\in A_j} a_iPe_i\|^2 \geq A\sum_{i\in A_j}|a_i|^2.$$

Theorem 5.3. *The Paving Conjecture is equivalent to Conjecture 5.2.*

Proof. Assume Conjecture 2.4 holds. Then there are universal constants $\epsilon>0$ and $r\in\mathbb{N}$ so that if P is a projection on ℓ_2^{2n} with $1/2's$ on the diagonal, since $I-P$ has this same property, there is a partition $\{A_j\}_{j=1}^r$ of $\{1,2,\cdots,2n\}$ so that for all $j=1,2,\cdots,r$ we have

$$\|Q_{A_j}(I-P)Q_{A_j}\|\leq 1-\epsilon.$$

Also, for every $f=\sum_{i=1}^{2n}a_ie_i\in\ell_2^{2n}$ and for all $j=1,2,\cdots,r$ we have

$$\begin{aligned}\langle Q_{A_j}(I-P)Q_{A_j}f,f\rangle &= \langle (I-P)Q_{A_j}f,Q_{A_j}f\rangle\\ &= \langle (I-P)Q_{A_j}f,(I-P)Q_{A_j}f\rangle\\ &= \|(I-P)Q_{A_j}f\|^2\\ &= \|\sum_{i\in A_j}a_i(I-P)e_i\|^2.\end{aligned}$$

Since $Q_{A_j}(I-P)Q_{A_j}$ is a self-adjoint operator we have

$$\sup_{\|f\|=1}\langle Q_{A_j}(I-P)Q_{A_j}f,f\rangle = \|Q_{A_j}(I-P)Q_{A_j}\|.$$

Since,

$$\|\sum_{i\in A_j}a_i(I-P)e_i\|^2+\|\sum_{i\in A_j}a_iPe_i\|^2=\sum_{i\in A_j}|a_i|^2,$$

we have

$$\|\sum_{i\in A_j}a_i(I-P)e_i\|^2\leq\|Q_{A_j}(I-P)Q_{A_j}\|\|f\|^2\leq(1-\epsilon)\sum_{i\in A_j}|a_i|^2.$$

Hence,

$$\|\sum_{i\in A_j}a_iPe_i\|^2\geq\epsilon\sum_{i\in A_j}|a_i|^2.$$

Hence, Conjecture 5.2 holds.

Conversely, if Conjecture 5.2 holds then there are universal $A>0$ and $r\in\mathbb{N}$ so that for any projection P with $1/2's$ on the diagonal, we can find a partition $\{A_j\}_{j=1}^r$ of $\{1,2,\cdots,2n\}$ so that for all $j=1,2,\cdots,r$ and all $f=\sum_{i\in A_j} a_i(I-P)e_i$ we have

$$\langle Q_{A_j}(I-P)Q_{A_j}f,f\rangle = \|\sum_{i\in A_j} a_i(I-P)e_i\|^2 \geq A\sum_{i\in A_j}|a_i|^2.$$

Since,

$$\sum_{i\in A_j}|a_i|^2 = \langle Q_{A_j}(I-P)Q_{A_j}f,f\rangle + \langle Q_{A_j}PQ_{A_j}f,f\rangle,$$

It follows that

$$\langle Q_{A_j}PQ_{A_j}f,f\rangle \leq (1-A)\sum_{i\in A_j}|a_i|^2 = (1-A)\|f\|^2.$$

Hence,

$$\|Q_{A_j}PQ_{A_j}\| \leq 1-A,$$

and so Conjecture 2.4 holds. □

It is known [23] that $r=2$ will not work in Conjecture 5.2. We remark that using the Rado-Horn Theorem [26, 27, 24] we can show that there is a subset $B\subset\{1,2,\cdots,2n\}$ with $|B|=n$ and both $\{Pe_i\}_{i\in B}$ and $\{Pe_i\}_{i\in B^c}$ are linearly independent. This implies there is an A which works for this set in Conjecture 5.2 - but the number A will depend on the dimension of the space.

It turns out we can also check just an upper inequality to get a problem equivalent to KS ([24], Theorem 4.6).

Conjecture 5.4. *There is a universal constant $1\leq D$ so that for all $T\in B(\ell_2^n)$ with $\|Te_i\|=1$ for all $i=1,2,\cdots,n$, there is an $r=r(\|T\|)$ and a partition $\{A_j\}_{j=1}^r$ of $\{1,2,\cdots,n\}$ so that for all $j=1,2,\cdots,r$ and all scalars $\{a_i\}_{i\in A_j}$*

$$\|\sum_{i\in A_j}a_iTe_i\|^2 \leq D\sum_{i\in A_j}|a_i|^2.$$

6. Kadison-Singer in Harmonic Analysis

In this section, we consider the Paving Conjecture for Toeplitz operators. This is the most worked on special case of the problem. Given $\phi\in L^\infty([0,1])$, the corresponding **Toeplitz operator** is $T_\phi : L^2[0,1]\to L^2[0,1]$, $T_\phi(f)=f\cdot\phi$. Note that some people would call these operators **Laurent Operators** or **Analytic Toeplitz Operators** and use "Toeplitz Operator" for operators defined on the span of the exponentials with positive integers in the exponent.

Throughout this section we will use the following notation.

Notation: If $I\subset\mathbb{Z}$, we let $S(I)$ denote the $L^2([0,1])$-closure of the span of the exponential functions with frequencies taken from I:

$$S(I) = \mathrm{cl}(\mathrm{span}\{\mathrm{e}^{2\pi int}\}_{n\in I}).$$

A deep and fundamental question in Harmonic Analysis is to understand the distribution of the norm of a function $f \in S(I)$. It is known (See [24]) if that if $[a,b] \subset [0,1]$ and $\epsilon > 0$, then there is a partition of $\mathbb{Z}$ into arithmetic progressions $A_j = \{nr + j\}_{n\in\mathbb{Z}}$, $0 \le j \le r-1$ so that for all $f \in S(A_j)$ we have

$$(1-\epsilon)(b-a)\|f\|^2 \le \|f \cdot \chi_{[a,b]}\|^2 \le (1+\epsilon)(b-a)\|f\|^2.$$

What this says is that the functions in $S(A_j)$ have their norms nearly uniformly distributed across $[a,b]$ and $[0,1] \setminus [a,b]$. The central question is whether such a result is true for arbitrary measurable subsets of $[0,1]$ (but it is known that the partitions can no longer be arithmetic progressions [16, 51, 52]).

If E is a measurable subset of $[0,1]$, let P_E denote the orthogonal projection of $L^2[0,1]$ onto $L^2(E)$, that is, $P_E(f) = f \cdot \chi_E$. The fundamental question here is then

Conjecture 6.1. *If $E \subset [0,1]$ is measurable and $\epsilon > 0$ is given, there is a partition $\{A_j\}_{j=1}^r$ of $\mathbb{Z}$ so that for all $j = 1, 2, \cdots, r$ and all $f \in S(A_j)$*

$$(1-\epsilon)|E|\|f\|^2 \le \|P_E(f)\|^2 \le (1+\epsilon)|E|\|f\|^2. \tag{6.1}$$

In [27] it is shown that Conjecture 6.1 is equivalent to the Paving Conjecture for Toeplitz operators. We do not know if the Paving Conjecture for Toeplitz operators is equivalent to Kadison-Singer. Recently [28] it was shown that either one of the inequalities in Conjecture 6.1 is equivalent to paving all Toeplitz operators. Despite the many deep results in the field of Harmonic Analysis, almost nothing is known about the distribution of the norms of functions coming from the span of a finite subset of the characters, except that this question has deep connections to Number Theory [16] (Also see [27, 24]). Actually, a significantly weaker conjecture is equivalent to the Paving Conjecture.

Conjecture 6.2. *There is a universal constant $0 < K$ so that for any measurable set $E \subset [0,1]$ there is a partition $\{A_j\}_{j=1}^r$ of $\mathbb{Z}$ so that for every $f \in S(A_j)$ we have $\|P_E f\|^2 \le K|E|\|f\|^2$.*

We also have a Feichtinger Conjecture for Toeplitz operators.

Definition 6.3. *We say the Toeplitz operator T_ϕ satisfies the Feichtinger Conjecture if there is a partition $\{A_j\}_{j=1}^r$ of $\mathbb{Z}$ so that $\{T_\phi e^{2\pi int}\}_{n\in A_j}$ is a Riesz basic sequence for every $j = 1, 2, \cdots, r$.*

At this time we do not know if the Feichtinger Conjecture for Toeplitz operators is equivalent to the Paving Conjecture for Toeplitz operators. It can be shown (See [24]) that the Feichtinger Conjecture for Toeplitz operators is equivalent to the corresponding statement for P_E. It is also shown in [24] that the Feichtinger Conjecture for Toeplitz operators is equivalent to:

Conjecture 6.4. *Suppose $E \subset [0,1]$ with $0 < |E|$. There is a partition $\{A_j\}_{j=1}^r$ of $\mathbb{Z}$ so that for all $j = 1, 2, \cdots, r$, P_E is an isomorphism of $S(A_j)$ onto its range.*

Next we look at the uniform versions of these properties.

Definition 6.5. *A Toeplitz operator T_g has the uniform paving property if for every $\epsilon > 0$, there is a $K \in \mathbb{N}$ so that if $A_k = \{nK + k\}_{n\in\mathbb{Z}}$ for $0 \le k \le K-1$ then*

$$\|P_{A_k}(T_g - D(T_g))P_{A_k}\| < \epsilon.$$

Definition 6.6. *A Toeplitz operator T_g has the uniform Feichtinger property if there is a $K \in \mathbb{N}$ so that if $A_k = \{nK + k\}_{n\in\mathbb{Z}}$ for $0 \le k \le K-1$, then $\{T_g e^{2\pi i n t}\}_{n\in A_k}$ is a Riesz basic sequence for all $k = 0, 1, \cdots, K-1$.*

Halpern, Kaftal and Weiss [51] made a detailed study of the uniform paving property and in particular they showed that T_ϕ is uniformly pavable if ϕ is a Riemann integrable function (See also [27, 24]).

Notation 6.7. *For all $g \in L^2[0,1]$, $K \in \mathbb{N}$ and any $0 \le k \le K-1$ we let*

$$g_k^K(t) = \sum_{n\in\mathbb{Z}} \langle g, e^{2\pi i(nK+k)t}\rangle e^{2\pi i(nK+k)t}.$$

Also,

$$g_K(t) = \frac{1}{K}\sum_{k=0}^{K-1} |g(t - \frac{k}{K})|^2.$$

The main theorems here are [24, 27]:

Theorem 6.8. *Let $g \in L^\infty[0,1]$ and T_g the Toeplitz operator of multiplication by g. The following are equivalent:*

(1) T_g has uniform PC.

(2) There is an increasing sequence of natural numbers $\{K_n\}$ so that

$$\lim_{n\to\infty} \frac{1}{K_n}\sum_{k=0}^{K_n-1} |g(t - \frac{k}{K_n})|^2 = \|g\|^2 \quad a.e.$$

uniformly over t. That is, for every $\epsilon > 0$ there is a $K \in \mathbb{N}$ so that

$$|\frac{1}{K}\sum_{k=0}^{K-1} |g(t - \frac{k}{K})|^2 - \|g\|^2| < \epsilon \quad a.e.$$

Theorem 6.9. *Let $g \in L^\infty([0,1])$ and T_g the Toeplitz operator of multiplication by g. The following are equivalent:*

(1) T_g has the uniform Feichtinger property.

(2) There is a natural number $K \in \mathbb{N}$ and an $\epsilon > 0$ so that

$$\frac{1}{K}\sum_{k=0}^{K-1} |g(t-\frac{k}{K})|^2 \geq \epsilon \quad a.e.$$

(3) There is an $\epsilon > 0$ and K measurable sets

$$E_k \subset [\frac{k}{K}, \frac{k+1}{K}], \quad \text{for all } 0 \leq k \leq K-1,$$

satisfying:

(a) The sets $\{E_k - \frac{k}{K}\}_{k=0}^{K-1}$ are disjoint in $[0, \frac{1}{K}]$.

(b) $\bigcup_{k=0}^{K-1}(E_k - k) = [0, \frac{1}{K}]$.

(c) $|g(t)| \geq \epsilon$ *on* $\bigcup_{k=0}^{K-1} E_k$.

It is known [11, 16, 24] that there are operators which fail the uniform Paving Property. It is also known [24] that there are operators with the uniform Fecihtinger Property which do not have the uniform Paving Property. But it can be shown:

Corollary 6.10. *For all Riemann integrable functions ϕ, the Toeplitz operator T_ϕ satisfies the uniform paving property. If $|g| \geq \epsilon > 0$ on an interval then g has the uniform Feichtinger property. Hence, if g is continuous at one point and is not zero at that point, then g has the uniform Feichtinger property.*

We end this section with a surprising theorem from [24] which gives an identity which holds for all $f \in L^2[0,1]$. It says that *pointwise*, any Fourier series can be divided into its subseries of arithmetic progressions so that the square sums of the functions given by the subseries spreads the norm nearly equally over the interval $[0,1]$ (Compare this to the discussion at the beginning of this section).

Theorem 6.11. *For any $g \in L^\infty([0,1])$ there is an increasing sequence of natural numbers $\{K_n\}_{n=1}^\infty$ so that*

$$\lim_{n\to\infty}\left(\sum_{k=0}^{K_n-1} |g_k^{K_n}(t)|^2\right)^{1/2} = \|g\|\chi_{[0,1]} \quad a.e.$$

7. Kadison-Singer in Time Frequency Analysis

Although the Fourier transform has been a major tool in analysis for over a century, it has a serious lacking for signal analysis in that it hides in its phases information concerning the moment of emission and duration of a signal. What was needed was a localized time-frequency representation which has this information encoded in it. In 1946 Gabor [38] filled this gap and formulated a fundamental approach to signal decomposition in terms of elementary signals. Gabor's method has become the paradigm for signal analysis in Engineering as well as its mathematical counterpart: Time-Frequency Analysis.

To build our elementary signals, we choose a **window function** $g \in L^2(\mathbb{R})$. For $x, y \in \mathbb{R}$ we define **modulation by x** and **translation by y** of g by:

$$M_x g(t) = \mathrm{e}^{2\pi \mathrm{i} x t} g(t), \quad T_y g(t) = g(t-y).$$

If $\Lambda \subset \mathbb{R} \times \mathbb{R}$ and $\{E_x T_y g\}_{(x,y)\in\Lambda}$ forms a frame for $L^2(\mathbb{R})$ we call this an (irregular) **Gabor frame**. Standard Gabor frames are the case where Λ is a lattice $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$ where $a, b > 0$ and $ab \le 1$. For an introduction to time-frequency analysis we recommend the excellent book of Grochenig [46].

It was in his work on time-frequency analysis that Feichtinger observed that all the Gabor frames he was working with could be decomposed into a finite union of Riesz basic sequences. This led him to formulate the Feichtinger Conjecture - which we now know is equivalent to KS. There is a significant amount of literature on the Feichtinger Conjecture for Gabor frames as well as wavelet frames and frames of translates [9, 10, 16]. It is known that Gabor frames over rational lattices [21] and Gabor frames whose window function is "localized" satisfy the Feichtinger Conjecture [9, 10, 47]. But the general case has defied solution.

Translates of a single function play a fundamental role in frame theory, time-frequency analysis, sampling theory and more [2, 16]. If $g \in L^2(\mathbb{R})$, $\lambda_n \in \mathbb{R}$ for $n \in \mathbb{Z}$ and $\{T_{\lambda_n} g\}_{n\in\mathbb{Z}}$ is a frame for its closed linear span, we call this a **frame of translates**. Although considerable effort has been invested in the Feichtinger Conjecture for frames of translates, little progress has been made. One exception is a surprising result from [22].

Theorem 7.1. *Let $I \subset \mathbb{Z}$ be bounded below, $a > 0$ and $g \in L^2(\mathbb{R})$. Then $\{T_{na} g\}_{n\in I}$ is a frame sequence if and only if it is a Riesz basic sequence.*

Our next theorem [24] will explain why the Feichtinger Conjecture for frames of translates, wavelet frames and Gabor frames has proven to be so difficult. This is due to the fact that this problem is equivalent to having all Toeplitz operators satisfy the Feichtinger Conjecture (See Conjecture 6.4).

Theorem 7.2. *The following are equivalent:*

(1) The Feichtinger Conjecture for frames of translates.

(2) For every $\phi \in L^2(\mathbb{R})$ and every $\{\lambda_n\}_{n\in\Lambda}$, if $\{\mathrm{e}^{2\pi \mathrm{i} \lambda_n t}\phi\}_{n\in\Lambda}$ is Bessel in $L^2(\mathbb{R})$, then it is a finite union of Riesz basic sequences.

(3) For every $0 \neq \phi \in L^2[0,1]$ and every $\Lambda \subset \mathbb{Z}$, if $\{\mathrm{e}^{2\pi \mathrm{i} n t}\phi\}_{n\in\Lambda}$ is a Bessel sequence then it is a finite union of Riesz basic sequences.

(4) For every $\phi \in L^2(\mathbb{R})$ and every $\{\lambda_n\}_{n\in\Lambda}$, if $\{e^{2\pi i\lambda_n t}\phi\}_{n\in\Lambda}$ is a frame sequence in $L^2(\mathbb{R})$, then it is a finite union of Riesz basic sequences.

(5) For every $\Lambda \subset \mathbb{Z}$ and every $0 \neq \phi \in L^2[0,1]$, if $\{e^{2\pi int}\phi\}_{n\in\Lambda}$ is a Bessel sequence, then it is a finite union of frame sequences.

We end this section with a result of Bownik and Speegle [16] which makes a connection between number theory and PC for Toeplitz operators. This is related to a possible generalization of van der Waerden's theorem [58, 59].

Definition 7.3. *Let $g : \mathbb{R}^2 \to \mathbb{R}$. We say that $I \subset \mathbb{Z}$ satisfies the $g(\ell, N)$-arithmetic progression condition if for every $\delta > 0$ there exists $M, N, \ell \in \mathbb{Z}$ such that*

(1) $g(\ell, N) < \delta$, and

(2) $\{M, M + \ell, \cdots, M + N\ell\} \subset I$.

Taking the Fourier transform through theorem 4.1.2 in [16] yields:

Theorem 7.4. *A positive solution to the Feichtinger Conjecture for Toeplitz operators implies there is a partition $\{A_j\}_{j=1}^r$ of $\mathbb{Z}$ so that each A_j fails the $g(\ell, N) = \ell N^{-1/2} log^3 N$ arithmetic progression condition.*

8. Kadison-Singer in Engineering

Frames have traditionally been used in signal processing because of their resilience to additive noise, resilience to quantization, numerical stability of reconstruction and the fact that they give greater freedom to capture important signal characteristics [40]. Recently, Goyal, Kovačević and Vetterli [40] (see also [43, 44, 41, 42]) proposed using the redundancy of frames to mitigate the losses in packet based transmission systems such as the internet. These systems transport packets of data from a "source" to a "recipient". These packets are sequences of information bits of a certain length surrounded by error-control, addressing and timing information that assure that the packet is delivered without errors. It accomplishes this by not delivering the packet if it contains errors. Failures here are due primarily to buffer overflows at intermediate nodes in the network. So to most users, the behavior of a packet network is not characterized by random loss but rather by *unpredictable transport time*. This is due to a protocol, invisible to the user, that retransmits lost or damaged packets. Retransmission of packets takes much longer than the original transmission and in many applications retransmission of lost packets is not feasible. If a lost packet is independent of the other transmitted data, then the information is truly lost. But if there are dependencies between transmitted packets, one could have partial or complete recovery despite losses. This leads us to consider using frames for encoding. But which frames? In this setting, when frame coefficients are lost we call them **erasures**. It was shown in

[39] that an equal norm frame minimizes mean-squared error in reconstruction with erasures if and only if it is tight. So a fundamental question is to identify the optimal classes of equal norm Parseval frames for doing reconstruction with erasures. Since the lower frame bound of a family of vectors determines the computational complexity of reconstruction, it is this constant we need to control. Formally, this is a max/min problem which looks like:

Problem 8.1. *Given natural numbers k, K find the class of equal norm Parseval frames $\{f_i\}_{i=1}^{Kn}$ in ℓ_2^n which maximize the minimum below:*

$$min\ \{A_J : J \subset \{1, 2, \cdots n\},\ |J| = k,\ A_J \text{ the lower frame bound of } \{f_i\}_{i \in J^c}\}.$$

This problem has proved to be very difficult. We only have a complete solution to the problem for two erasures [13, 25, 54]. Recently, [57] examined the erasure problem for equiangular circular frames using some deep results from number theory. It was hoped that some special cases of the problem would be more tractable and serve as a starting point for the classification since the frames we are looking for are contained in this class.

Conjecture 8.2. *There exists an $\epsilon > 0$ so that for large K, for all n and all equal norm Parseval frames $\{f_i\}_{i=1}^{Kn}$ for ℓ_2^n, there is a $J \subset \{1, 2, \cdots, Kn\}$ so that both $\{f_i\}_{i \in J}$ and $\{f_i\}_{i \in J^c}$ have lower frame bounds which are greater than ϵ.*

The ideal situation would be for Conjecture 8.2 to hold for all $K \geq 2$. It was shown in [27] that Conjecture 8.2 implies KS. We do not know if the converse is true.

If we are going to be able to erase arbitrary k-element subsets of our frame, then the frame must be a union of erasure sets. So a generalization of Conjecture 8.2 which is a class containing the class given in Problem 8.1 is

Conjecture 8.3. *There exists $\epsilon > 0$ and a natural number r so that for all large K and all equal norm Parseval frames $\{f_i\}_{i=1}^{Kn}$ in ℓ_2^n there is a partition $\{A_j\}_{j=1}^{r}$ of $\{1, 2, \cdots, Kn\}$ so that for all $j = 1, 2, \cdots, r$ the Bessel bound of $\{f_i\}_{i \in A_j}$ is $\leq 1 - \epsilon$.*

In [27] it was shown that Conjecture 8.3 is equivalent to the Paving Conjecture.

In [24, 27] it is shown that the Paving Conjecture also arises naturally in the famous **Cocktail Party Problem** in speech recognition technology. Finally, [24, 27] give a relationship between Paving and the recent significant discoveries surrounding sparse solutions to vastly underdetermined systems of linear equations initiated by Donoho and Huo [17, 18, 19, 20, 33, 34, 35, 37, 60].

9. Kadison-Singer in Algebraic Geometry

In this section we discuss Conjectures 2.1 and 2.4 in the context of the geometry of the Grassmannian, as well as some related questions that arise from the geometric point of view.

9.1. **Grassmannians of planes in $\mathbb{C}^n$.** We begin by recalling the definition of the Grassmannian. The (complex) Grassmannian, $G(k,n)$, is by definition, the set of k-dimensional linear subspaces of $\mathbb{C}^n$. This set can be given the structure of a $k \times (n-k)$-dimensional non-singular projective variety, which means that it is a compact manifold. For example, the Grassmannian $G(1,n)$ is the classical projective space of lines in $\mathbb{C}^n$.

Grassmannians naturally arise in many situations in algebraic geometry and have been extensively studied. See for example the book of Griffiths and Harris [45]. Of particular interest are *Schubert subvarieties.* In the context of the KS problem we will consider the following special kind of Schubert varieties in $G(k,n)$. Let L_0 be a fixed linear subspace and set

$$\sigma_{L_0} = \{L \in G(k,n) | L \cap L_0 \neq 0\}.$$

A basic fact is that σ_{L_0} is an algebraic subvariety of $G(k,n)$. If the dimension of L_0 is $> n-k$ then $\sigma_{L_0} = G(k,n)$. However, if $\dim L_0 \leq n-k$ then σ_{L_0} is a *proper subvariety.* The significance of this fact is that when σ_{L_0} is not all of $G(k,n)$ it has dimension less than $k \times (n-k)$, the dimension of $G(k,n)$. The significance of this is that the complement of σ_{L_0} is dense, since it is open in the Zariski topology.

9.2. **Grassmannians and orthogonal projections.** Fix an orthonormal basis $\{e_i\}_{i=1}^N$ of l_2^N. A linear subspace is uniquely determined by its orthogonal projection. Thus the set of orthogonal projections of rank k is identified with the Grassmannian $G(k,n)$. Since P is a projection, $\operatorname{rank} P = \operatorname{trace} P$, so $G(k,n)$ is the same as projections with trace k. The map $G(k,n) \to \mathbb{R}^n$ which takes a projection to its diagonal is called the *moment map.* A theorem of Horn [55] implies that in this case the image of the moment map is the convex hull of the $\binom{n}{k}$ vectors which are permutations of the vector $(1, 1, \cdots 1, 0, \cdots, 0)$ whose first k entries are 1's. (The result of Horn was subsequently generalized to the theorem of the convexity of the moment map proved by Atiyah [8] and independently Guillimen and Sternberg [48]. The monograph [49] gives an introduction to these ideas.)

Hence for any vector $(\delta_1, \cdots, \delta_n) \in \mathbb{R}^n_{\geq 0}$ with $\sum \delta_i = k$ there is a non-empty family of rank P projections with diagonal $(\delta_1, \cdots, \delta_n)$. Because the moment map is differentiable (in fact real algebraic) and the Grassmannian is compact the set of projections with given diagonal is a real algebraic compact submanifold of $G(k,n)$.

9.3. **Paving projections and open sets in the Grassmannians.** Let P be an orthogonal projection with image L. If L' is a fixed linear subspace with orthogonal projection Q then the eigenvalues of PQP are all in $[0,1]$. Observe that 1 is an eigenvalue of PQP if and only $L' \cap L \neq 0$. Hence $||PQP|| < 1$ if and only if $L' \cap L = 0$.

Proposition 9.1. *The set of rank k projection operators P on l_2^n which are (r,ϵ)-pavable for some $\epsilon > 0$ (depending on P) is a (possibly empty) Zariski open set in $G(k,n)$.*

Proof. Let P be a projection with image L. For fixed r, the projection P is not (r,ϵ) pavable for any ϵ if and only if for every partition $\{A_i\}_{i=1}^r$ of $\{1,\cdots,n\}$, $[L] \in \sigma_i$ for some i, where σ_i denotes the subvariety of $G(k,n)$ defined by the condition $\{[L] \cap \operatorname{Image} Q_{A_i} \neq 0\}$.

Thus P is (r,ϵ)-pavable for some $\epsilon > 0$ if and only $[L]$ is in the complement of the union of a finite number of subvarieties of $G(k,n)$. This is a Zariski open set. □

Proposition 9.2. *The Zariski open set of rank k projections P on l_2^n which are (r,ϵ) pavable is empty if $k > n(1-1/r)$.*

Proof. If $\{A_i\}_{i=1}^r$ is a partition then one of the A_i's must have at least n/r elements. Suppose $|A_j| \geq n/r$. Let $L_j = \operatorname{Image} Q_{A_j}$. Then $\dim L_j \geq n/r$, so if L is a linear subspace of dimension $k > n(1-1/r)$ then $L \cap L_j \neq 0$. □

9.4. **Projections with fixed diagonal.** Using the Rado-Horn theorem, a partial converse of Proposition 9.2 can proved. This result is an unused part of a project of Casazza, Edidin, Kalra and Paulson [23].

Proposition 9.3. [23] *If $\delta \leq 1 - 1/r$ then for every natural number n there is an $\epsilon_n > 0$ such that every orthogonal projection P on l_2^n with diagonal entries all $\leq \delta$ is $(r, 1-\epsilon_n)$-pavable.*

We will not prove this proposition but instead prove a generalization of it in Proposition 9.6 below.

Proposition 9.3 leads to a number of interesting problems about the pavability of projections in $\mathbb{R}^n$ with a given diagonal. Since the pavability of a given projection is unchanged by permuting rows and columns we will assume that the projections we consider have monotonically decreasing diagonals.

Problem 9.4. *Give necessary and sufficient conditions on a vector $\mathbf{d} = (\delta_1,\cdots,\delta_n)$ with $1 \geq \delta_1 \geq \delta_2 \geq \cdots \geq \delta_n \geq 0$ so that every orthogonal projection with diagonal $\mathbf{d}$ is $(r, 1-\epsilon_{\mathbf{d}})$-pavable for some $\epsilon_{\mathbf{d}} > 0$.*

Proposition 9.3 says that $\delta_1 \leq 1 - 1/r$ is sufficient. However, this condition is clearly not necessary. Consider for example projections of rank 1. If P is an orthogonal projection with image the linear subspace spanned by a unit vector v, then P is 2-pavable if and only v is not a standard basis vector. By taking v to be arbitrarily close to a standard basis vector we can produce 2-pavable projections such that δ_1 is arbitrarily close to 1. On the other hand, for projections of rank $k = n(1 - 1/r)$ the condition of Proposition 9.3 is necessary as there are projections with diagonal entries $\delta_i \geq 1 - 1/r + \epsilon$ which are not r-pavable.

We will now prove a generalization of Proposition 9.3. For this we will need to recall the Rado-Horn Theorem (See [26] and its references).

Theorem 9.5 (Rado-Horn). *Let I be a finite or countable index set and let $\{f_i\}_{i \in I}$ be a collection of vectors in a vector space. There is a partition $\{A_j\}_{j=1}^r$ such that for each $j = 1, 2, \cdots, r$, $\{f_i\}_{i \in A_j}$ is linearly independent if and only if for all finite $J \subset I$*

$$\frac{|J|}{\dim \operatorname{span} \ \{f_i\}_{i \in J}} \leq r. \tag{9.1}$$

We can now prove the generalization of Proposition 9.3.

Proposition 9.6. *Given a monotnically decreasing sequence $1 > \delta_1 \geq \delta_2 \geq \cdots \geq \delta_n \geq 0$ with $\sum_{i=1}^n \delta_i \in \mathbb{N}$ satisfying the inequalities*

$$\sum_{i=1}^{rm+1} \delta_i < (r-1)m + 1 \tag{9.2}$$

for all integers $m \geq 0$, there is an $\epsilon_d > 0$ such that every orthogonal projection P with diagonal $\mathbf{d} = (\delta_1, \cdots, \delta_n)$ is $(r, 1 - \epsilon_d)$-pavable.

Proof. We proceed by way of contradiction. If the result fails, then there are projections $\{P_i\}_{i=1}^\infty$ on l_2^n with diagonal $\mathbf{d}$ so that P_i is not $(r, 1 - \frac{1}{i})$-pavable. Since the family of projections on l_2^n with diagonal $\mathbf{d}$ is compact in the operator topology, it follows that $\{P_i\}_{i=1}^\infty$ has a convergent subsequence converging to a projection P with diagonal $\mathbf{d}$. It follows that P is not (r, ϵ)-pavable for any $\epsilon > 0$. We will obtain a contradiction by showing that there exists an $\epsilon > 0$ so that P is (r, ϵ)-pavable for some $\epsilon > 0$. For this, we will check that the conditions of the Rado-Horn Theorem hold for the vectors $\{(I - P)e_i\}_{i=1}^n$. Fix $J \subset \{1, 2, \cdots, n\}$ with $|J| = rm + k$ and $0 \leq k < r$. Let P_J be the orthogonal projection onto the span of $\{(I - P)e_i\}_{i \in J}$. Since $\{1 - \delta_i\}_{i=1}^n$ is decreasing we have

$$\begin{aligned}\dim\text{span}\ \{(I-P)e_i\}_{i\in J} &= \sum_{i=1}^{n}\|P_J(I-P)e_i\|^2\\ &\geq \sum_{i\in J}\|(I-P)e_i\|^2\\ &= \sum_{i\in J}(1-\delta_i)\\ &\geq \sum_{i=1}^{rm+k}(1-\delta_i)\\ &= rm+k-\sum_{i=1}^{rm+1}\delta_i-\sum_{i=rm+2}^{rm+k}\delta_i\\ &> rm+k-(r-1)m-(k-1)=m.\end{aligned}$$

Since the left hand side of the above inequality is an integer, it follows that

$$\dim\text{span}\ \{(I-P)e_i\}_{i\in J}\geq m+1\geq m+\frac{k}{r}=\frac{rm+k}{r}=\frac{|J|}{r}.$$

By the Rado-Horn Theorem, there is a partition $\{A_j\}_{j=1}^r$ of $\{1,2,\cdots,n\}$ so that for all $1\leq j\leq r$ the family $\{(I-P)e_i\}_{i\in A_j}$ is linearly independent. It follows that, for any $j=1,2,\cdots,r$, $(I-P)Q_{A_j}$ does not have zero as an eigenvalue. Since

$$\langle Q_{A_j}(I-P)Q_{A_j}f,f\rangle=\langle (I-P)Q_{A_j}f,Q_{A_j}f\rangle=\|(I-P)Q_{A_j}f\|^2,$$

it follows that $(Q_{A_j}(I-P)Q_{A_j})$ does not have zero as an eigenvalue and hence $Q_{A_j}PQ_{A_j}$ does not have one as an eigenvalue. That is, P is (r,ϵ)-pavable for some $\epsilon>0$. This contradiction completes the proof of the theorem. □

Problem 9.7. *Are the inequalities* (9.2) *necessary conditions?*

There is good evidence that the answer to Problem 9.7 is yes because of the following proposition.

Proposition 9.8. *Let* $\mathbf{d}=(\delta_1,\cdots,\delta_n)$ *with* $0\leq\delta_i\leq 1$ *and* $\sum_{i=1}^n\delta_i\in\mathbb{N}$*. If*

$$\sum_{i=1}^{rm+1}\delta_i=(r-1)m+1$$

for some integer $m\geq 0$ *then there exists an orthogonal projection with diagonal* $\mathbf{d}$ *which is not* $(r,1-\epsilon)$ *pavable for any* $\epsilon>0$.

Proof. First we can construct a projection P_1 with diagonal entries $\{\delta_i\}_{i=1}^{rm+1}$ with P_1 projecting onto an $[(r-1)m+1]$-dimensional Hilbert space $\mathbb{H}_1$. Next construct a projection P_2 with diagonal entries $\{\delta_i\}_{i=rm+2}^{n}$ on a Hilbert space $\mathbb{H}_2$. Let $P = P_1 \oplus P_2$ be the projection on $\mathbb{H}_1 \oplus \mathbb{H}_2$. If $\{A_i\}_{i=1}^{r}$ is a partition of $\{1, 2, \cdots, n\}$ then one of these sets, say A_k, contains at least $m+1$ elements from $\{1, 2, \cdots, rm+1\}$. Let $B_k = A_k \cap \{1, 2, \cdots, rm+1\}$. Since dim $(I_1 - P_1) = rm+1-((r-1)m+1) = m$, it follows that $\{(I_1-P_1)e_i\}_{i\in B_k}$ is linearly dependent and so $Q_{B_k}(I_1 - P_1)Q_{B_k}$ has zero as an eigenvalue. So P_1 (and hence P) has one as an eigenvalue. That is, P is not $(r, 1-\epsilon)$-pavable for any $\epsilon > 0$. $\square$

A solution to Kadison-Singer problem would be given by computing for fixed diagonal $\mathbf{d} = (\delta_1, \cdots, \delta_n)$ the "worst paving constant" $\epsilon_{\mathbf{d}}$. Obviously this problem is more difficult than Kadison-Singer since Conjectures 2.1 and 2.4 only ask whether there is a δ such that if $\delta_i \leq \delta$ then $\epsilon_{\mathbf{d}} \to 1-\epsilon$ as $n \to \infty$. However, a more tractable question is about the monotonicity of the worst paving constant.

Again consider the set of diagonals $\mathbf{d} = (\delta_1, \cdots, \delta_n)$ with $1 \geq \delta_1, \geq \cdots \geq \delta_n \geq 0$. Define a partial order on the set of diagonals as follows: If $\mathbf{d}' = (\delta_1', \cdots, \delta_n')$ then $\mathbf{d}' \leq \mathbf{d}$ iff $\delta_i' \leq \delta_i$ for $i = 1, \cdots, n$.

Problem 9.9. *Is the worst paving constant $\epsilon_{\mathbf{d}}$ monotonic for the partial ordering on diagonals; i.e. if $\mathbf{d}' \leq d$ then $\epsilon_{\mathbf{d}'} \leq \epsilon_{\mathbf{d}}$?*

Fix a vector $\mathbf{d} = (\delta_1, \cdots, \cdots \delta_n) \in \mathbb{R}^n$ with $1 \geq \delta_1 \geq \cdots, \geq \delta_n \geq 0$ and $\sum \delta_i = k$ with $k \leq n$ a positive integer. Let $G_{\mathbf{d}}(k,n) \subset G(k,n)$ be the real algebraic subvariety of projections with diagonal $\mathbf{d}$. Consider the continuous function $\phi_r \colon G_{\mathbf{d}}(k,n) \to [0,1]$,

$$P \mapsto \min\{\max_{1\leq i\leq r}\{||Q_{A_i}PQ_{A_i}|| : \{A_i\}_{i=1}^{r} \text{ is a partition of } \{1, 2, \cdots, n\}\}.$$

If we assume that every projection in $G_{\mathbf{d}}(k,n)$ is r-pavable then the image of ϕ_r does not contain 1. Since $G_{\mathbf{d}}(k,n)$ is compact the image is compact so ϕ_r attains its maximum $1-\epsilon_{\mathbf{d}}$ for some $\epsilon_{\mathbf{d}} > 0$.

Problem 9.10. *Describe the geometry locus, $\phi_r^{-1}(1-\epsilon_{\mathbf{d}})$, of orthogonal projections with the worst paving.*

This locus is invariant under the conjugation action of the group $\mathbb{T}^{n-1}$ of unitary diagonal matrices, since conjugation by a diagonal unitary matrix leaves the diagonal unchanged. Some basic questions are whether this locus is connected, smooth or real algebraic. While the function ϕ_r is not obviously differentiable, or real algebraic it is plausible that $\phi_r^{-1}(1-\epsilon_{\mathbf{d},r})$ may have a nicer structure as the following example shows.

Example 9.11. *When $n = 4$ and $\delta = (1/2, 1/2, 1/2, 1/2)$ then a simple argument shows that the locus of worst paved projections consists of the $\mathbb{T}^{n-1}$ orbit* *of projections conjugate to* $(1/2)I - (1/2\sqrt{3})C_4$ *where*

$$C_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}$$

is the unique 4×4 *conference matrix. Hence this locus is a connected real algebraic submanifold of* $G(2,4)$.

If C_n is an $n \times n$ conference matrix then $I_n - (1/2\sqrt{n})C_n$ is an orthogonal projection with 1/2 on the diagonal. However, the results of [23] show that these projections are strongly 2-pavable while arbitrary projections with 1/2 on the diagonal are not. Thus for n sufficiently large these projectoins are not the worst pavable.

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Department of Mathematics, University of Missouri, Columbia, MO 65211-4100
E-mail address: pete,edidin@math.missouri.edu