SQUARE-FREE MONOMIAL IDEALS AND HYPERGRAPHS

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ABSTRACT. These are notes on the author's talk given at the workshop on Integral Closure, Multiplier Ideals and Cores, AIM, December 2006.

1. INTRODUCTION

Let K be a field and $R = K[x_1, ..., x_n]$ a polynomial ring over K.

Let G be a simple, undirected graph on the set of vertices $[1, n] = \{1, \ldots, n\}$. The edge ideal of G is defined to be the ideal

$$I(G) := (x_i x_j | \ \{i, j\} \in G).$$

This notion was introduced by Villareal.

Let $I = I(G)$. Simis, Vasconcelos and Villareal showed that $I^{(k)} = I^k \ \forall \ k \geq 0$ if and only if G is a bipartite graph, where $I^{(k)}$ denotes the k-th symbolic power of I. In particular, the equality between symbolic and ordinary powers holds for the edge ideal of any subgraph of G if G is bipartite.

The edge ideals of graphs are in one-to-one correspondence with the ideals generated by square free monomials of degree two. To study arbitrary squarefree monomial ideals we need to consider hypergraphs.

A *clutter* Δ on the set of vertices $[1, n]$ is a collection of subsets of $[1, n]$ called edges such that there is no containment of edges.

Definition 1.1. The *edge ideal* of Δ is defined to be the ideal

$$I(\Delta) := (x_{i_1} ... x_{i_r} | \ \{i_1, ..., i_r\} \in \Delta).$$

The edge ideals of clutters are in one-to-one correspondence with the square free monomial ideals. There is an other way to associate a clutter with a squarefree monomial ideal.

Definition 1.2. The ideal

$$I^*(\Delta) = \bigcap_{\{i_1, ..., i_r\} \in \Delta} (x_{i_1}, \ldots, x_{i_r})$$

is called the *cover ideal* of Δ.

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Recall that a cover of Δ is a subset of $[1, n]$ which meets every edge of Δ. The minimal generators of $I^*(\Delta)$ are in one-to-one correspondence with the minimal covers of Δ.

Let Δ^* denote the clutter of the minimal covers of Δ. Then Δ^* is called the *blocker* or *tranversal* of Δ. We always have $(\Delta^*)^* = \Delta$ and hence $I^*(\Delta) = I(\Delta^*)$. The ideal $I^*(\Delta)$ is also called the Alexander dual of $I(\Delta)$.

2. Cover ideals

Definition 2.1. Let $k \in \mathbb{N}$ and $C = (c_1, \ldots, c_n) \in \mathbb{N}^n$. Then C is called a k*–cover* of Δ if

$$\sum_{i \in F} c_i \geq k \ \ \forall\ F \in \Delta.$$

If we think of C as a multiset which consists of c_i copies of i, $i = 1, ..., n$, then a k–cover is just a multiset of vertices which meet every edge at least k times.

Definition 2.2. Let t be an indeterminate. The algebra

$$A(\Delta) = K[x^C t^k \mid C \text{ is a } k\text{-cover of } \Delta,\ k \geq 0] \subseteq R[t]$$

is called the *vertex cover algebra* of Δ.

Let $I = I^*(\Delta)$. It is obvious that $x^C = x_1^{c_1} \ldots x_n^{c_n} \in I^{(k)}$ if and only if C is a k–cover. Therefore, $A(\Delta) = \oplus_{k \geq 0} I^{(k)} t^k$, the symbolic Rees algebra of I. Thus, $I^{(k)} = I^k$ for all $k \geq 0$ if and only if $A(\Delta)$ is a standard graded algebra over R.

In order to generalize the result of Simis, Vasconcelos and Villareal we need to consider hypergraphs which generalize bipartite graphs.

Definition 2.3. A *cycle* of Δ is an alternating sequence of the form

$$v_1, F_1, v_2, F_2, \ldots, v_r, F_r, v_{r+1} = v_1,$$

where $v_1, ..., v_r$ and $F_1, ..., F_r$ are distinct vertices and edges and $x_i, v_{i+1} \in F_i\ \forall\ i = 1, \ldots, n$. The cycle is called *special* if $F_i \cap \{v_1, \ldots, v_r\} = \{v_i, v_{i+1}\}$. We call Δ a *balanced* clutter if Δ has no odd special cycle of length greater or equal to 3.

Notice that a graph G is balanced if and only if G is bipartite.

Theorem 2.4. [Herzog-Hibi-Trung-Zheng] $A(\Gamma)$ *is standard graded for all subclutters* $\Gamma \subset \Delta$ *if and only if* Δ *is balanced.*

3. Edge ideal

Let $I = I(\Delta)$ now be the edge ideal of Δ. By the above result, $I^{(k)} = I^k$ for all $k \geq 0$ if Δ^* is balanced. The question now is whether we can describe this equality directly in terms of Δ.

Let $\overline{I^k}$ denote the integral closure of I^k. Since $I^k \subseteq \overline{I^k} \subseteq I^{(k)}$, we can break up the equality into two parts $I^k \subseteq \overline{I^k}$ and $\overline{I^k} = I^{(k)}$. All these equalities can be expressed in combinatorial terms.

Let $\Delta = \{F_1, \ldots, \ldots F_m\}$. Let $M = (a_{ij})$ be the vertex-edge incidence matrix of Δ, i.e.

$$a_{ij} = \begin{cases} 0, & i \notin F_j, \\ 1, & i \in F_j. \end{cases}$$

For all vectors $C \in \mathbb{N}^n$ we define

$$\begin{aligned} \tau(C) &:= \min\{\ A \cdot \mathcal{C} \mid A \in \mathbb{N}^n,\ M^T \cdot A \geq \underline{1},\ \}, \\ \tau^*(C) &:= \min\{\ A \cdot \mathcal{C} \mid A \in \mathbb{R}^n_+,\ M^T \cdot A \geq \underline{1},\ \}, \\ \nu(C) &:= \max\{B \cdot \underline{1} \mid B \in \mathbb{N}^m,\ M \cdot B \leq C\ \}, \\ \nu^*(C) &:= \max\{B \cdot \underline{1} \mid B \in \mathbb{R}^m_+,\ M \cdot B \leq C\ \}, \end{aligned}$$

where $\underline{1}$ denotes the vector $(1, ..., 1) \in \mathbb{N}^m$. Then

$$\nu(C) \leq \nu^*(C) = \tau^*(C) \leq \tau(C),$$

where the middle equalilty follows from the duality of Linear Programming.

Lemma 3.1.

(1) $x^C \in I^k \Leftrightarrow \nu(C) \geq k$.
(2) $x^C \in \overline{I^k} \Leftrightarrow \nu^*(C) \geq k$.
(3) $x^C \in I^{(k)} \Leftrightarrow \tau(C) \geq k$.

Definition 3.2.

(1) Δ is said to have the integer rounding property if $\nu(C) = [\nu^*(C)]$ for all $C \in \mathbb{N}^n$.
(2) Δ is called Fulkersonian if $\tau(C) = \tau^*(C)$ for all $C \in \mathbb{N}^n$.
(3) Δ is called Mengerian if $\nu(C) = \tau(C)$ for all $C \in \mathbb{N}^n$.

Lemma 3.1 gives easy proofs for the following results.

Theorem 3.3.

(1) $I^k = \overline{I^k}$ *for all* $k \geq 0$ *if and only if* Δ *has the rounding property.*
(2) $\overline{I^k} = I^{(k)}$ *for all* $k \geq 0$ *if and only if* Δ *is a Fulkersonian hypergraph* [Trung, Villarreal].
(3) $I^k = I^{(k)}$ *for all* $k \geq 0$ *if and only if* Δ *is a Mengerian hypergraph* [Herzog-Hibi-Trung-Zheng, Villarreal].

Unlike balanced hypergraphs, there are no known characterizations of the above classes of hypergraphs by forbidden structure.

4. The König property

Definition 4.1. We call the following invariants

$$\tau(\Delta) = \min\{c_1 + \cdots + c_n \mid C \text{ is a minimal cover of } \Delta\},$$
$$\nu(\Delta) = \max\{k \mid \text{there exist } k \text{ disjoint edges of } \Delta\}$$

the *blocking number* and the *matching number* of Δ, respectively. If $\nu(\Delta) = \tau(\Delta)$, Δ is said to have the *König property*.

Notice that $\nu(\Delta) = \nu(C)$ and $\tau(\Delta) = \tau(C)$, where $C = (1, ..., 1)$.

Let V_1, V_2 be two arbitrary disjoint subsets of the set of vertices of Δ. We define a clutter Γ on the set of vertices $V \setminus (V_1 \cup V_2)$ whose edges are the subsets of V of the form $F \setminus V_1$, where $F \in \Delta$ and $F \cap V_2 = \emptyset$. Clearly, $I(\Gamma)$ is obtained from $I(\Delta)$ by setting $x_i = 1$ for $i \in V_1$ and $x_i = 0$ for $i \in V_2$.

Definition 4.2. Such a clutter Γ is called a *minor* of $I(\Delta)$.

Conjecture 4.3. [Conforti-Cornuejols] *Δ is a Mengerian hypergraph if and only if all minors of Δ has the König property.*

Note that the implication "only if" is trivial. It is also known that Δ is a Fulkerson hypergraph if all minors of Δ has the König property. Therefore, it remains to show that Δ has the rounding property if all minors of Δ has the König property,

Let $I = I(\Delta)$. Then $\tau(\Delta) = \operatorname{ht}(I)$ and $\nu(\Delta)$ is the maximal length of a regular sequence of monomials in I. To settle the conjecture we have to show that $I^k = \overline{I^k}$ for all $k \geq 0$ if all ideals J obtained from I by setting some variables equal 0,1 contain a regular sequence of monomials of length $\operatorname{ht}(J)$. Roughly speaking, we have to consider ideals obtained from I by adding some variables and localize at some variables.

On the other hand, $I^k = \overline{I^k}$ for all $k \geq 0$ means that the Rees algebra $R(I)$ is normal. Since $R(I)$ is normal if and only if it satisfies Serre conditions R_1 and S_2, using induction we need only to show that depth $R(I) \geq 2$ under the assumption of the conjecture.