Directed Section and Projection Functions

(based on joint work with Paul Goodey)

1. Classical Uniqueness Theorems

a) $K, M \subset \mathbb{R}^d$ c.s. star bodies,
$u \in S^{d-1}$

$$V_{d-1}(K \cap u^\perp) = V_{d-1}(M \cap u^\perp) \quad \forall u$$

$$\Longrightarrow \quad K = M$$

Proof: $V_{d-1}(K \cap u^\perp) = c\left(R\rho_K^{d-1}\right)(u)$

$\Longrightarrow R\rho_K^{d-1} = R\rho_M^{d-1} \Rightarrow \rho_K = \rho_M \Rightarrow K = M$

due to the injectivity of the spherical Radon transform R (on even f.)

$$R : f \longmapsto Rf = \int_{\cdot^\perp \cap S^{d-1}} f(x)\, dx .$$

b) $K, M \subset \mathbb{R}^d$ c.s. convex bodies (always of full dimension), $u \in S^{d-1}$

$$V_{d-1}(K|u^\perp) = V_{d-1}(M|u^\perp) \quad \forall u$$

$$\Rightarrow K = M$$

Proof: $V_{d-1}(K|u^\perp) = c\,(C\,S_{d-1}(K,\cdot))(u)$

$$\Rightarrow C\,S_{d-1}(K,\cdot) = C\,S_{d-1}(M,\cdot)$$

$$\Rightarrow S_{d-1}(K,\cdot) = S_{d-1}(M,\cdot)$$

$$\Rightarrow K = M$$

due to the injectivity of the Cosine transform C (on even μ)

$$C : \mu \longmapsto f = C\mu = \int_{S^{d-1}} |\langle \cdot, x\rangle| \, d\mu(x)$$

and the Minkowski theorem.

The two situations show a lot of similarities, but there are also essential differences:

In b), replace V_{d-1} by intrinsic volume V_j, $1 \le j \le d-1$:

$$V_j(K|u^\perp) = V_j(M|u^\perp) \ \forall u \in S^{d-1}$$

$$\Rightarrow \quad K = M$$

Proof: $V_j(K|u^\perp) = c\big(C\, S_j(K,\cdot)\big)(u)$

$\Rightarrow$ .... (with Alexandrov-Fenchel th.)

In a), replace V_{d-1} by dual intrinsic volume $\tilde{V}_j$, $1 \le j \le d-1$ $\Rightarrow$ no change in problem.

(If V_{d-1} is replaced by surface area, ... $\to$ open problem (Richard Gardner))

2. Lower Dimensions

a) K, M c.s. star bodies in $\mathbb{R}^d$,
$L \in \mathcal{L}^d_k$ (Grassmanian of k-spaces)

$$V_k(K \cap L) = V_k(M \cap L) \quad \forall L \in \mathcal{L}^d_k$$
$$\Rightarrow K = L$$

Proof: Simple, since 1) implies

$V_k(K \cap L) = V_k(M \cap L) \; \forall L \Rightarrow$

$\rho_K = \rho_M$ on each $(k+1)$-space

$\Rightarrow \rho_K = \rho_M \Rightarrow K = M.$

b) K, M c.s. convex bodies, $L \in \mathcal{L}_k^d$

$$V_k(K|L) = V_k(M|L) \ \forall L \Rightarrow K = M$$

Proof: Uses Cauchy-Kubota f.

$$V_k(K|L) = V_k(M,L) \overset{\forall L}{\Longrightarrow}$$

$$V_k(K|u^\perp) = V_k(M|u^\perp) \ \forall u$$

$$\Longrightarrow \quad K = M$$

(above remark)

This shows also that k can be replaced by $j \in \{1, \dots, k\}$.

3) Non-symmetric Sets

Section function $u \mapsto V_{d-1}(K \cap u^\perp)$ yields only symmetrized set

$K \tilde{+} (-K)$ ← radial addition

Projection function $u \mapsto V_{d-1}(K | u^\perp)$ yields Minkowski sum $K + (-K)$

Different attempts for generalizations:

- sections through two points or non-central sections (Larman)
- two functionals (mean width + Steiner point) (Schneider)
- sum of two functions (projection volume + illumination function) (mentioned by Richard G.)

4) Directed Section Functions

$K \subset \mathbb{R}^d$ star body, $L \in \mathcal{L}^d_k$, $u \in L$, $\|u\| = 1$.

$$s_k(K, L, u) = V_k(K \cap L \cap u^+)$$

$$u^+ = \{ x \in \mathbb{R}^d : \langle x, u \rangle \geq 0 \}$$

Thm (Goodey-W.): K, M star bodies

$$s_k(K, \cdot) = s_k(M, \cdot) \Rightarrow K = M$$

Proof: Injectivity of Radon tr. on even f. yields even part of ρ_K.

Injectivity of Cap transform on odd f. yields odd part of ρ_K.

Body seems to be overdetermined by $s_k(K, \cdot)$!?

5) Average Section Function

$$\bar{s}_k(K,u) = \int_{\substack{L \in \mathcal{L}^d_k \\ u \in L}} s_k(K,L,u)\, dL$$

Does $\bar{s}_k(K,\cdot)$ determine K ?

Eq.: Is the operator

$$A^d_k : \rho^k(K,\cdot) \longmapsto \bar{s}_k(K,\cdot)$$

injective?

A^d_k is a continuous, linear + intertwining operator on $C(S^{d-1})$

$\Rightarrow$ eigenspaces are spherical harm

$\Rightarrow$ (injective $\Leftrightarrow$ all mult. $a_{d,k,n} \neq 0$)

Thm. For $2 \le k \le d-1$,

$$(A_k^d f)(u) = c \int_{u^\perp \cap S^{d-1}} f(x)\left(1 - \langle x,u\rangle^2\right)^{(k-d)/2} dx$$

and

$$a_{d,k,n} = \int_0^1 (1-t^2)^{(k-3)/2} P_n^d(t)\, dt,$$

Legendre polynomial

$n = 0, 1, 2, \ldots$

Even multipliers (direct formula):

$$a_{d,k,2n} = \frac{(1/2)_n((d-k)/2)_n}{(k/2)_n((d-1)/2)_n} \cdot \frac{\Gamma(1/2)\Gamma((k-1)/2)}{2\Gamma(k/2)} > 0,$$

$$n = 0, 1, 2, \ldots$$

with Pochhammer symbol

$$(x)_n = x(x+1)\cdots(x+n-1), \quad (x)_0 = 1.$$

For the odd multipliers, earlier results of Paul Goodey (based on partial integration and recursion formulas for Legendre polynomials) yield a

recursion formula for the odd multipliers:

$$
\begin{aligned}
(2n+d+1)(2n+d+2)(2n+k+3)(4n+d+2)a_{d,k,2n+5} \\
= (2n+2)(2n+3)(2n+d-k+1)(4n+d+6)a_{d,k,2n+1} \\
+ (4n+d+4)\Big\{4(d-2k+2)n^2 + 2(d+4)(d-2k+2)n \\
+ 4d^2 + 5d - 7dk - 2k + 6\Big\}a_{d,k,2n+3}
\end{aligned}
$$

with initial values

$$
\begin{aligned}
a_{d,k,1} &= \frac{1}{k-1}, \\
a_{d,k,3} &= \frac{2d-3k+1}{(d-1)(k^2-1)}.
\end{aligned}
$$

Consequences for odd multipliers ($d \neq 4$):

- for $k \leq \frac{d+2}{2}$, we have $a_{d,k,2n+1} > 0$ (injectivity)
- for $k > \frac{2d+1}{3}$, the $a_{d,k,2n+1}$ alternate in sign (injectivity)
- for $k = \frac{2d+1}{3}$, we have $a_{d,k,3} = 0$ (non-injectivity)

Theorem. *Let $K, M \subset \mathbb{R}^d$ be star bodies and $2 \le k \le \frac{d}{2} + 1$ or $\frac{2d+1}{3} < k \le d - 1$, for $d \ne 4$ (respectively $k = 2$, for $d = 4$). If*

$$\overline{s}_k(K, \cdot) = \overline{s}_k(M, \cdot),$$

then $K = M$.

On the other hand, for $i = 1, 2, \ldots,$ there are star bodies $K \ne M$ in $\mathbb{R}^{3i+1}$ with

$$\overline{s}_{2i+1}(K, \cdot) = \overline{s}_{2i+1}(M, \cdot).$$

There is also a corresponding Stability Theorem in case of convex bodies K, M.

6) Directed Projection Functions

7) Average Projection Functions

Since $V_j(K) = c S_j(K, S^{d-1})$

If $\overline{v}_{j,k}(K,u)$ is the average of $v_{j,k}(K,L,u)$ over all $L \in \mathcal{L}^d_k$, $L \ni u$,

for $j = 1, \dots, d-1$,

j-th surface area measure

would define $\overline{v}_{1,k}(K,\cdot)$ gives rise to a linear operator (in an apparent way):

$v_{j,k}(K,L,u) = c\, S'_j(K|L,\, u^+ \cap L \cap S^{d-1})$

for $L \in \mathcal{L}^d_k$, almost all $u \in L \cap S^{d-1}$.

$$\overline{v}_{1,k}(K,u) = c \int_{\substack{L \in \mathcal{L}^d_k \\ u \in L}} S'_1(K|L,\, u^+ \cap L)\, dL$$

Th $1 \le j \le k-1 \le d-2$

$v_{j,k}(K,\cdot) = v_{j,k}(M,\cdot)$

$\Rightarrow$ convex bodies K, M are translates

$\underbrace{\left(\frac{1}{k-1}\Delta_L + 1\right) h_K\big|_L}$

Proof: Cap transform & Cosine transf. projection formula of Schneider for surface area measures

$$(M^d_k g)(u) = \int_{\substack{L \in \mathcal{L}^d_k \\ u \in L}} \int_{L \cap u^+} \left[\left(\frac{1}{k-1}\Delta_L + 1\right) g\right](x)\, dx\, dL$$

The multipliers of M_k^d

- For even functions f, we have $M_k^d f = A_k^d f$, hence the even multipliers $\gamma_{d,k,2n}$ of M_k^d are equal to $a_{d,k,2n}$, and hence positive.

- For the odd multipliers, we hope again for a recursion formula!

By definition (with Legendre polynomials to obtain special spherical harmonics),

$$\gamma_{d,k,n} = \int_{S^{d-1}\cap L\cap u^+} \Box_L P_n^d(\langle u, \cdot\rangle)(x)dx \qquad \text{(for fixed } L \in \mathcal{L}_k^d, u \in L\text{)}.$$

Using the connection coefficients $c_{n,m}^{d,k}$ between Legendre polynomials in dimensions d and k,

$$c_{n,m}^{d,k} = \frac{(2(n-2m)+k-2)n!}{(k-2)m!(n-2m)!} \frac{\left(\frac{d-2}{2}\right)_{n-m} \left(\frac{d-k}{2}\right)_m (k-2)_{n-2m}}{\left(\frac{k}{2}\right)_{n-m} (d-2)_n}$$

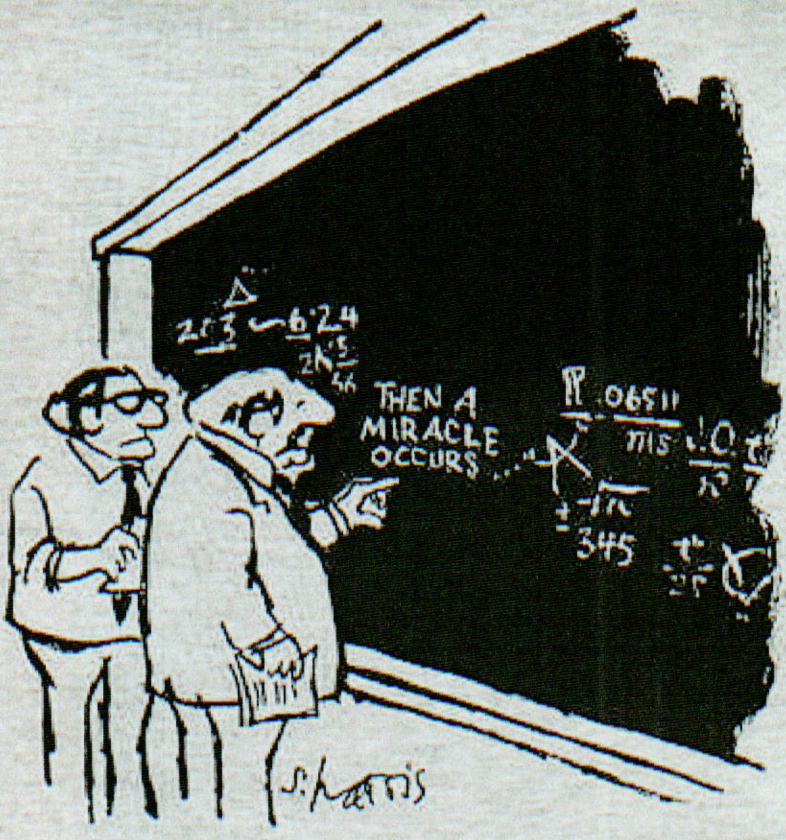
(and the well-known eigenvalues of $\Box_L$), one gets

$$\gamma_{d,k,2n+1} = \sum_{m=0}^{n-1} F(m,n)$$

with

$$F(m,n) := (-1)^{n-m+1} \frac{2(n-m)(2n-2m+k)}{k-1} \frac{\left(\frac{1}{2}\right)_{n-m}}{\left(\frac{k+1}{2}\right)_{n-m}} c^{d,k}_{2n+1,m}$$

To see here a recursion formula, we need a miracle!



"I THINK YOU SHOULD BE MORE EXPLICIT
HERE IN STEP TWO."

...miracle... $\Rightarrow$ Recursion formula

$$\begin{aligned} p(n)\gamma_{d,k,2n+5} &= a(n)\gamma_{d,k,2n+1} \\ &\quad + (n+2)(2n+d)(4n+d+4)e(n)\gamma_{d,k,2n+3} \end{aligned}$$

where

$$\begin{aligned} p(n) &= (n+1)(2n+d)(2n+d+1)(2n+d+2) \\ &\qquad \cdot (2n+k+3)(4n+d+2) \\ a(n) &= 2(n+1)(n+2)(2n+3)(2n+d+2) \\ &\qquad \cdot (2n+d-k+1)(4n+d+6) \\ e(n) &= 4(d-2k-2)n^2 + 2(d+4)(d-2k-2)n \\ &\qquad + (d+2)(2d-5k-3) \end{aligned}$$

initial values are (apart from $\gamma_{d,k,1} = 0$)

$$\gamma_{d,k,3} = \frac{2(d+2)}{(d-1)(k+1)}$$

and

$$\gamma_{d,k,5} = \frac{4(d+4)(2d-5k-3)}{(d^2-1)(k+1)(k+3)}$$

Theorem. *Let* $K, M \subset \mathbb{R}^d$ *be convex bodies and* $2 \leq k \leq \frac{2d-3}{5}$ *or* $\frac{d-2}{2} < k \leq d-1$. *If*

$$\overline{v}_{1,k}(K,\cdot) = \overline{v}_{1,k}(M,\cdot),$$

then K, M *are translations of one another.*

On the other hand, for $i = 1, 2, \ldots,$ *there are convex bodies* $K \neq M$ *in* $\mathbb{R}^{5i+4}$ *with*

$$\overline{v}_{1,2i+1}(K,\cdot) = \overline{v}_{1,2i+1}(M,\cdot).$$

The miracle (Zeilberger's Algorithm)

The recursion formula is found with Zeilberger's algorithm (computer package).

Recall that

$$\gamma_{d,k,2n+1} = \sum_{m=0}^{n-1} F(m,n)$$

with

$$F(m,n) := (-1)^{n-m+1} \frac{2(n-m)(2n-2m+k)}{k-1} \frac{\left(\frac{1}{2}\right)_{n-m}}{\left(\frac{k+1}{2}\right)_{n-m}} c_{2n+1,m}^{d,k}$$

Zeilberger's algorithm produces a rational function $R(m, n)$ and expressions $a(n), e(n), p(n)$ such that

$$\begin{aligned} &a(n)F(m,n) + (n+2)(2n+d)(4n+d+4)e(n)F(m,n+1) \\ &\quad - p(n)F(m,n+2) = F(m+1,n)R(m+1,n) - F(m,n)R(m,n) \end{aligned}$$

Summing over $m = 0, \ldots, n$, the right-hand side gives 0, hence the left-hand side yields the recursion formula.

$R(m,n)$ is a product $r(m,n)s(m,n)$ with

$$
\begin{aligned}
r(m,n) :=& \frac{m(2n-2m+1)(1+n)(2+n)(3+2n)}{(n-m)(1-m+n)(3-2m+2n)(5-2m+2n)(k-4m+4n)(d+2n)} \\
&\times \frac{(d-2m+4n)(d+2n)(1+d+2n)(2+d+2n)}{(d+2n-1)(1+d+2n)(2+d+2n)(1+k-2m+2n)(3+k-2m+2n)} \\
&\times \frac{(2n+k-2m-1)(2+k-2m+2n)}{(2+k-2m+4n)(4+k-2m+4n)(6+k-2m+4n)(8+k-2m+4n)}
\end{aligned}
$$

$$
\begin{aligned}
s(m,n) :=& -64(4+d+4n)t_6(n)m^6 + 32t_5(n)m^5 - 16t_4(n)m^4 + 8t_3(n)m^3 \\
&- 4t_2(n)m^2 + 2t_1(n)m - t_0(n)
\end{aligned}
$$

Here, the t_i are polynomials in n (depending on the fixed values d,k) and the main task of the algorithm is to find these.

$k \to m$

Consider

$$R(k,n) = r(k,n)s(k,n)$$

with

$$\begin{aligned} r(k,n) = {} & \frac{k(2n-2k+1)(1+n)(2+n)(3+2n)}{(n-k)(1-k+n)(3-2k+2n)(5-2k+2n)(j-4k+4n)} \\ & \times \frac{(d-2k+4n)(d+2n)(1+d+2n)(2+d+2n)}{(d+2n-1)(d+2n)(1+d+2n)(2+d+2n)(1+j-2k+2n)(3+j-2k+2n)} \\ & \times \frac{(2n+j-2k-1)(2+j-2k+2n)}{(2+j-2k+4n)(4+j-2k+4n)(6+j-2k+4n)(8+j-2k+4n)} \end{aligned}$$

and

$$\begin{aligned} s(k,n) = {} & -64(4+d+4n)t_6(n)k^6 + 32t_5(n)k^5 - 16t_4(n)k^4 + 8t_3(n)k^3 \\ & - 4t_2(n)k^2 + 2t_1(n)k - t_0(k,n) \end{aligned}$$

with

$$t_6(n) = -12 - 6d + 2d^2 + d^3 + 8j + 2dj - d^2j - 32n + 8dn + 4d^2n - 16n^2 + 8dn^2,$$

$$\begin{aligned} t_5(n) = {} & -1056 - 1040d - 20d^2 + 170d^3 + 26d^4 + 864j + 354dj - 43d^2j + 6d^3j + 4d^4j + 128j^2 \\ & + 54dj^2 - 13d^2j^2 - 4d^3j^2 - 4640n - 1728dn + 824d^2n + 328d^3n + 16d^4n + 1424jn \\ & + 432djn + 20d^2jn + 22d^3jn + 160j^2n + 8dj^2n - 20d^2j^2n - 7040n^2 + 256dn^2 + 1200d^2n^2 \\ & + 136d^3n^2 + 560jn^2 + 328djn^2 + 60d^2jn^2 + 32j^2n^2 - 16dj^2n^2 - 4480n^3 + 1408dn^3 \\ & + 416d^2n^3 + 64jn^3 + 160djn^3 - 1024n^4 + 512dn^4 + 64jn^4, \end{aligned}$$

$$\begin{aligned} t_4(n) = {} & -9936 - 11636d - 838d^2 + 1796d^3 + 274d^4 + 9064j + 2936dj - 272d^2j + 439d^3j \\ & + 88d^4j + 2764j^2 + 1060dj^2 - 235d^2j^2 - 25d^3j^2 + 6d^4j^2 + 172j^3 + 56dj^3 - 27d^2j^3 \\ & - 6d^3j^3 - 50256n - 29144dn + 7312d^2n + 4708d^3n + 348d^4n + 20952jn + 6168djn \\ & + 1922d^2jn + 948d^3jn + 54d^4jn + 5252j^2n + 858dj^2n - 278d^2j^2n + 14d^3j^2n + 220j^3n \\ & - 38dj^3n - 36d^2j^3n - 95424n^2 - 15664dn^2 + 17672d^2n^2 + 3768d^3n^2 + 108d^4n^2 \\ & + 15856jn^2 + 8512djn^2 + 3392d^2jn^2 + 420d^3jn^2 + 2952j^2n^2 + 168dj^2n^2 - 20d^2j^2n^2 \\ & + 32j^3n^2 - 52dj^3n^2 - 86848n^3 + 11744dn^3 + 12608d^2n^3 + 944d^3n^3 + 5312jn^3 + 7136djn^3 \\ & + 1320d^2jn^3 + 624j^2n^3 + 120dj^2n^3 - 16j^3n^3 - 38400n^4 + 13440dn^4 + 2880d^2n^4 + 2240jn^4 \\ & + 2240djn^4 + 160j^2n^4 - 6656n^5 + 3328dn^5 + 896jn^5, \end{aligned}$$

$j \to k$

$$
\begin{aligned}
t_3(n) = {} & -50880 - 66000d - 7700d^2 + 9210d^3 + 1460d^4 + 47312j + 11666dj - 7d^2j + 4494d^3j \\
& + 750d^4j + 22512j^2 + 8220dj^2 - 1300d^2j^2 + 325d^3j^2 + 108d^4j^2 + 2728j^3 + 836dj^3 \\
& - 382d^2j^3 - 51d^3j^3 + 4d^4j^3 + 88j^4 + 14dj^4 - 23d^2j^4 - 4d^3j^4 - 288704n - 217216dn \\
& + 26720d^2n + 31320d^3n + 2876d^4n + 142880jn + 39240djn + 22144d^2jn \\
& + 11956d^3jn + 944d^4jn + 58368j^2n + 13560dj^2n - 176d^2j^2n + 912d^3j^2n \\
& + 66d^4j^2n + 5128j^3n - 12dj^3n - 572d^2j^3n - 10d^3j^3n + 88j^4n \\
& - 68dj^4n - 28d^2j^4n - 648704n^2 - 225696dn^2 + 111664d^2n^2 + 37320d^3n^2 \\
& + 1840d^4n^2 + 162000jn^2 + 81008djn^2 + 47712d^2jn^2 + 9736d^3jn^2 + 292d^4jn^2 \\
& + 54288j^2n^2 + 10928dj^2n^2 + 2328d^2j^2n^2 + 452d^3j^2n^2 + 2624j^3n^2 - 792dj^3n^2 \\
& - 156d^2j^3n^2 - 32j^4n^2 - 56dj^4n^2 - 752384n^3 - 29184dn^3 + 127168d^2n^3 \\
& + 18752d^3n^3 + 384d^4n^3 + 93056jn^3 + 96192djn^3 + 34048d^2jn^3 + 2496d^3jn^3 \\
& + 24000j^2n^3 + 7424dj^2n^3 + 1280d^2j^2n^3 + 224j^3n^3 - 208dj^3n^3 - 32j^4n^3 \\
& - 480768n^4 + 94848dn^4 + 59904d^2n^4 + 3392d^3n^4 + 41216jn^4 + 55712djn^4 \\
& + 8048d^2jn^4 + 7360j^2n^4 + 2560dj^2n^4 - 161792n^5 + 60416dn^5 + 10240d^2n^5 \\
& + 19456jn^5 + 12160djn^5 + 1792j^2n^5 - 22528n^6 + 11264dn^6 + 4864jn^6,
\end{aligned}
$$

$$
\begin{aligned}
t_2(n) = {} & -147552 - 202744d - 32812d^2 + 23974d^3 + 4069d^4 + 129096j + 23084dj + 5892d^2j + 19396d^3j \\
& + 3042d^4j + 86540j^2 + 31782dj^2 - 1364d^2j^2 + 3588d^3j^2 + 699d^4j^2 + 15488j^3 + 4692dj^3 \\
& - 1744d^2j^3 + 3d^3j^3 + 56d^4j^3 + 960j^4 + 138dj^4 - 241d^2j^4 - 33d^3j^4 + d^4j^4 + 12j^5 \\
& - 4dj^5 - 7d^2j^5 - d^3j^5 - 927712n - 826016dn + 21728d^2n + 103492d^3n + 11152d^4n \\
& + 490552jn + 128168djn + 109306d^2jn + 63988d^3jn + 5898d^4jn + 290004j^2n \\
& + 85534dj^2n + 13788d^2j^2n + 9910d^3j^2n + 870d^4j^2n + 39432j^3n + 4132dj^3n \\
& - 2644d^2j^3n + 278d^3j^3n + 34d^4j^3n + 1496j^4n - 640dj^4n - 390d^2j^4n \\
& - 14d^3j^4n - 4j^5n - 30dj^5n - 8d^2j^5n - 2393792n^2 - 1224816dn^2 + 321288d^2n^2 \\
& + 167008d^3n^2 + 11088d^4n^2 + 757904jn^2 + 368608djn^2 + 279168d^2jn^2 + 75212d^3jn^2 \\
& + 3736d^4jn^2 + 382952j^2n^2 + 113176dj^2n^2 + 35376d^2j^2n^2 + 8324d^3j^2n^2 + 268d^4j^2n^2 \\
& + 34648j^3n^2 - 940dj^3n^2 - 624d^2j^3n^2 + 184d^3j^3n^2 + 184j^4n^2 - 940dj^4n^2 \\
& - 132d^2j^4n^2 - 32j^5n^2 - 20dj^5n^2 - 3334464n^3 - 685600dn^3 + 553920d^2n^3 \\
& + 127792d^3n^3 + 4768d^4n^3 + 638112jn^3 + 566592djn^3 + 283680d^2jn^3 + 37760d^3jn^3 \\
& + 776d^4jn^3 + 264160j^2n^3 + 99840dj^2n^3 + 27648d^2j^2n^3 + 2208d^3j^2n^3 + 12720j^3n^3 \\
& - 336dj^3n^3 + 288d^2j^3n^3 - 512j^4n^3 - 280dj^4n^3 - 16j^5n^3 - 2737664n^4 \\
& + 116096dn^4 + 412480d^2n^4 + 46880d^3n^4 + 752d^4n^4 + 365696jn^4 + 465024djn^4 \\
& + 129792d^2jn^4 + 6896d^3jn^4 + 115584j^2n^4 + 53248dj^2n^4 + 7072d^2j^2n^4 + 2720j^3n^4 \\
& + 560dj^3n^4 - 160j^4n^4 - 1336320n^5 + 320000dn^5 + 145152d^2n^5 + 6656d^3n^5 \\
& + 181632jn^5 + 193152djn^5 + 22368d^2jn^5 + 38208j^2n^5 + 11936dj^2n^5 + 704j^3n^5 \\
& - 361472n^6 + 141056dn^6 + 19840d^2n^6 + 70400jn^6 + 32000djn^6 + 7296j^2n^6 \\
& - 41984n^7 + 20992dn^7 + 12800jn^7,
\end{aligned}
$$

$$
\begin{aligned}
t_1(n) = (&-225984 - 322480d - 69080d^2 + 28780d^3 + 5474d^4 + 173968j + 20632dj + 19876d^2j + 38178d^3j \\
&+ 5826d^4j + 156880j^2 + 61844dj^2 + 7482d^2j^2 + 11726d^3j^2 + 1918d^4j^2 + 36908j^3 + 11816dj^3 \\
&- 2487d^2j^3 + 904d^3j^3 + 244d^4j^3 + 3296j^4 + 448dj^4 - 766d^2j^4 - 63d^3j^4 + 10d^4j^4 \\
&+ 84j^5 - 28dj^5 - 49d^2j^5 - 7d^3j^5 - 1570880n - 1577056dn - 95664d^2n + 159568d^3n \\
&+ 19988d^4n + 812624jn + 211240djn + 250652d^2jn + 153026d^3jn + 15524d^4jn \\
&+ 658720j^2n + 242232dj^2n + 72348d^2j^2n + 38108d^3j^2n + 3648d^4j^2n + 122100j^3n \\
&+ 24382dj^3n - 1808d^2j^3n + 2770d^3j^3n + 300d^4j^3n + 7256j^4n - 1732dj^4n \\
&- 1548d^2j^4n - 28d^3j^4n + 6d^4j^4n + 20j^5n - 226dj^5n - 84d^2j^5n \\
&- 4d^3j^5n - 4590592n^2 - 3015552dn^2 + 340624d^2n^2 + 336888d^3n^2 + 27792d^4n^2 \\
&+ 1628352jn^2 + 813632djn^2 + 734616d^2jn^2 + 236104d^3jn^2 + 15140d^4jn^2 + 1150640j^2n^2 \\
&+ 443944dj^2n^2 + 168552d^2j^2n^2 + 44556d^3j^2n^2 + 2284d^4j^2n^2 + 155784j^3n^2 + 21352dj^3n^2 \\
&+ 4828d^2j^3n^2 + 2492d^3j^3n^2 + 92d^4j^3n^2 + 3944j^4n^2 - 4052dj^4n^2 - 900d^2j^4n^2 \\
&+ 12d^3j^4n^2 - 240j^5n^2 - 260dj^5n^2 - 32d^2j^5n^2 - 7444608n^3 - 2705152dn^3 \\
&+ 1023392d^2n^3 + 355232d^3n^3 + 18608d^4n^3 + 1863936jn^3 + 1544000djn^3 + 958144d^2jn^3 \\
&+ 176272d^3jn^3 + 6432d^4jn^3 + 1104224j^2n^3 + 492256dj^2n^3 + 168096d^2j^2n^3 \\
&+ 22496d^3j^2n^3 + 472d^4j^2n^3 + 99136j^3n^3 + 16432dj^3n^3 + 6288d^2j^3n^3 + 704d^3j^3n^3 \\
&- 1136j^4n^3 - 2336dj^4n^3 - 120d^2j^4n^3 - 240j^5n^3 - 80dj^5n^3 - 7397888n^4 \\
&- 836864dn^4 + 1124864d^2n^4 + 200384d^3n^4 + 6048d^4n^4 + 1414656jn^4 + 1619200djn^4 \\
&+ 641792d^2jn^4 + 64064d^3jn^4 + 1008d^4jn^4 + 666624j^2n^4 + 340224dj^2n^4 + 77952d^2j^2n^4 \\
&+ 4176d^3j^2n^4 + 37824j^3n^4 + 11296dj^3n^4 + 2016d^2j^3n^4 - 1312j^4n^4 - 336dj^4n^4 \\
&- 64j^5n^4 - 4655104n^5 + 454656dn^5 + 623872d^2n^5 + 57984d^3n^5 + 768d^4n^5 \\
&+ 812800jn^5 + 964736djn^5 + 217152d^2jn^5 + 9120d^3jn^5 + 282624j^2n^5 + 133120dj^2n^5 \\
&+ 13888d^2j^2n^5 + 11712j^3n^5 + 3296dj^3n^5 - 192j^4n^5 - 1826816n^6 + 492544dn^6 \\
&+ 175360d^2n^6 + 6784d^3n^6 + 370688jn^6 + 307712djn^6 + 29568d^2jn^6 + 84224j^2n^6 \\
&+ 22144dj^2n^6 + 2432j^3n^6 - 411648n^7 + 165888dn^7 + 19968d^2n^7 + 114688jn^7 \\
&+ 40960djn^7 + 12800j^2n^7 - 40960n^8 + 20480dn^8 + 16384jn^8,
\end{aligned}
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\begin{aligned}
t_0(n) = &-140544 - 208704d - 58272d^2 + 10704d^3 + 2616d^4 + 90624j + 5424dj + 19032d^2j + 27876d^3j \\
&+ 4194d^4j + 107568j^2 + 48448dj^2 + 15064d^2j^2 + 12652d^3j^2 + 1893d^4j^2 + 30936j^3 + 11296dj^3 \\
&+ 178d^2j^3 + 1804d^3j^3 + 336d^4j^3 + 3456j^4 + 536dj^4 - 652d^2j^4 + 14d^3j^4 + 21d^4j^4 \\
&+ 120j^5 - 40dj^5 - 70d^2j^5 - 10d^3j^5 - 1084800n - 1203680dn - 177712d^2n + 84920d^3n \\
&+ 12868d^4n + 510560jn + 136816djn + 214840d^2jn + 133604d^3jn + 14484d^4jn \\
&+ 553488j^2n + 254352dj^2n + 106360d^2j^2n + 48678d^3j^2n + 4880d^4j^2n + 129632j^3n \\
&+ 39580dj^3n + 7418d^2j^3n + 5900d^3j^3n + 626d^4j^3n + 10416j^4n - 808dj^4n \\
&- 1576d^2j^4n + 108d^3j^4n + 26d^4j^4n + 128j^5n - 356dj^5n - 178d^2j^5n \\
&- 14d^3j^5n - 3575680n^2 - 2827168dn^2 - 22832d^2n^2 + 239176d^3n^2 + 24084d^4n^2 \\
&+ 1284000jn^2 + 689328djn^2 + 713240d^2jn^2 + 257700d^3jn^2 + 19408d^4jn^2 + 1227232j^2n^2 \\
&+ 597704dj^2n^2 + 260460d^2j^2n^2 + 72928d^3j^2n^2 + 4656d^4j^2n^2 + 223512j^3n^2 + 63052dj^3n^2 \\
&+ 20496d^2j^3n^2 + 6956d^3j^3n^2 + 388d^4j^3n^2 + 10448j^4n^2 - 4072dj^4n^2 - 1208d^2j^4n^2 \\
&+ 136d^3j^4n^2 + 8d^4j^4n^2 - 328j^5n^2 - 636dj^5n^2 - 140d^2j^5n^2 - 4d^3j^5n^2 \\
&- 6651136n^3 - 3409984dn^3 + 564608d^2n^3 + 336368d^3n^3 + 22656d^4n^3 + 1908544jn^3 \\
&+ 1580672djn^3 + 1133488d^2jn^3 + 256928d^3jn^3 + 12688d^4jn^3 + 1542144j^2n^3 + 811536dj^2n^3 \\
&+ 312704d^2j^2n^3 + 53552d^3j^2n^3 + 1952d^4j^2n^3 + 207312j^3n^3 + 63392dj^3n^3 \\
&+ 22480d^2j^3n^3 + 3584d^3j^3n^3 + 80d^4j^3n^3 + 2784j^4n^3 - 3872dj^4n^3 - 208d^2j^4n^3 \\
&+ 48d^3j^4n^3 - 624j^5n^3 - 400dj^5n^3 - 32d^2j^5n^3 - 7754240n^4 - 2072576dn^4 \\
&+ 994944d^2n^4 + 264256d^3n^4 + 11488d^4n^4 + 1884160jn^4 + 2022016djn^4 + 995648d^2jn^4 \\
&+ 140320d^3jn^4 + 4064d^4jn^4 + 1229056j^2n^4 + 687936dj^2n^4 + 202016d^2j^2n^4 \\
&+ 19392d^3j^2n^4 + 304d^4j^2n^4 + 117824j^3n^4 + 44128dj^3n^4 + 11360d^2j^3n^4 + 688d^3j^3n^4 \\
&- 1344j^4n^4 - 1120dj^4n^4 + 64d^2j^4n^4 - 352j^5n^4 - 80dj^5n^4 - 5927936n^5 \\
&- 301568dn^5 + 809728d^2n^5 + 118272d^3n^5 + 3008d^4n^5 + 1330688jn^5 + 1551104djn^5 \\
&+ 498432d^2jn^5 + 40000d^3jn^5 + 512d^4jn^5 + 667392j^2n^5 + 363648dj^2n^5 + 67968d^2j^2n^5 \\
&+ 2784d^3j^2n^5 + 47232j^3n^5 + 18880dj^3n^5 + 2208d^2j^3n^5 - 640j^4n^5 - 64j^5n^5 - 3000320n^6 \\
&+ 409088dn^6 + 358656d^2n^6 + 28288d^3n^6 + 320d^4n^6 + 705024jn^6 + 715520djn^6 \\
&+ 134016d^2jn^6 + 4672d^3jn^6 + 256512j^2n^6 + 110208dj^2n^6 + 9408d^2j^2n^6 + 14208j^3n^6 \\
&+ 3520dj^3n^6 - 978944n^7 + 285696dn^7 + 83968d^2n^7 + 2816d^3n^7 + 275456jn^7 \\
&+ 184320djn^7 + 15104d^2jn^7 + 65536j^2n^7 + 14592dj^2n^7 + 2304j^3n^7 - 188416n^8 \\
&+ 77824dn^8 + 8192d^2n^8 + 69632jn^8 + 20480djn^8 + 8192j^2n^8 - 16384n^9 + 8192dn^9 + 8192jn^9.
\end{aligned}
$$