Sums of similar convex bodies and spherical harmonics

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Basic operations on the space $\mathcal{K}^n$ of convex bodies in $\mathbb{R}^n$:

Minkowski addition:

$$K + L := \{x + y : x \in K,\, y \in L\}, \qquad K, L \in \mathcal{K}^n,$$

dilatation:

$$\alpha K := \{\alpha x : \boldsymbol{x} \in K\}, \qquad K \in \mathcal{K}^n,\, \alpha \geq 0.$$

The support function

$$h_K(u) := \max\{\langle u, x\rangle : x \in K\} \qquad u \in \mathbb{R}^n,$$

has the nice property that

$$h_{K+L} = h_K + h_L, \qquad h_{\alpha K} = \alpha h_K.$$

Moreover, for the Hausdorff metric δ,

$$\delta(K, L) = \max\{|h_K(u) - h_L(u)| : u \in S^{n-1}\}.$$

The simplest non-trivial convex body is a segment.

Support function of a segment S with center 0:

$$h_S(u) = |\langle u, v \rangle| \alpha \qquad \text{with } v \in S^{n-1},\ \alpha > 0.$$

A zonotope is a sum of finitely many segments.
Support function of a zonotope Z with center 0:

$$h_Z(u) = \sum_{i=1}^{k} |\langle u, v_i \rangle| \alpha_i \qquad \text{with } v_i \in S^{n-1},\ \alpha_i > 0.$$

A zonoid is a limit of zonotopes.

Support function of a zonoid Z with center 0:

$$h_Z(u) = \int_{S^{n-1}} |\langle u, v \rangle| \, \rho(dv)$$

with a finite Borel measure ρ.

A generalized zonoid K with center 0 has support function

$$h_K(u) = \int_{S^{n-1}} |\langle u, v \rangle| \, \rho(dv)$$

with a finite signed Borel measure ρ.

K generalized zonoid $\Leftrightarrow$ $h_K = h_{Z_2} - h_{Z_1}$ with zonoids Z_1, Z_2

$\Leftrightarrow$ $K + Z_1 = Z_2$ with zonoids Z_1, Z_2

Let $\mathcal{K}_s^n$ denote the set of centrally symmetric convex bodies.

(a) The zonoids are nowhere dense in $\mathcal{K}_s^n$.
(b) The generalized zonoids are dense in $\mathcal{K}_s^n$.

Fact (b) has often been useful in the investigation of centrally symmetric convex bodies.

Reminder of the proof of (b):

For given $K \in \mathcal{K}_s^n$ with center 0, try to solve the integral equation

$$h_K(u) = \int_{S^{n-1}} |\langle u, v \rangle| \, f(v) \, d\sigma(v)$$

($\sigma =$ spherical Lebesgue measure) with an integrable function f.

If h_K is sufficiently smooth, a continuous solution f exists (by expansions in spherical harmonics). An even solution is unique.

Question: (to get rid of the central symmetry)

Can the segment S be replaced by a non-symmetric convex body?
(to obtain a dense class in $\mathcal{K}^n$ instead of $\mathcal{K}^n_s$)

In other words:

Suppose we have only one convex body B at our hands and want to produce other convex bodies from it by taking Minkowski linear combinations of congruent copies of B, limits, and differences.

How big a class of convex bodies can we obtain?

Definitions:

Minkowski class: a subset of $\mathcal{K}^n$ that is closed

- in the Hausdorff metric,
- under Minkowski linear combinations,
- under translations.

Let G be a subgroup of $GL(n)$, for example $SO(n)$.

The Minkowski class $\mathcal{M}$ is G-invariant if
$K \in \mathcal{M} \Rightarrow gK \in \mathcal{M}$ for all $g \in G$.

If $B \in \mathcal{K}^n$ and $G \subset GL(n)$ are given, $\mathcal{M}_{B,G}$ is defined as the smallest G-invariant Minkowski class containing B.

Examples: If S is a segment and B is a ball, then

$$\mathcal{M}_{S,SO(n)} = \mathcal{M}_{S,GL(n)} = \{\text{zonoids}\},$$

$$\mathcal{M}_{B,SO(n)} = \{\text{balls}\},$$

$$\mathcal{M}_{B,GL(n)} = \{\text{zonoids}\}.$$

K is called an $\mathcal{M}$-body if $K \in \mathcal{M}$.

K is called a generalized $\mathcal{M}$-body if $K + M_1 = M_2$ with $\mathcal{M}$-bodies M_1, M_2.

Fact: For $B \in \mathcal{K}^n$, the class $\mathcal{M}_{B,GL(n)}$ is nowhere dense in $\mathcal{K}^n$.

What about generalized $\mathcal{M}_{B,GL(n)}$-bodies?

Theorem (Sch. 1996) *Let $T \subset \mathbb{R}^n$ be a triangle with an irrational angle. Then the set of generalized $\mathcal{M}_{T,SO(n)}$-bodies is dense in $\mathcal{K}^n$.*

Theorem (Alesker 2003) *Let K be a non-symmetric convex body. Then the set of generalized $\mathcal{M}_{K,GL(n)}$-bodies is dense in $\mathcal{K}^n$.*

Alesker's proof uses representation theory for the group $GL(n)$.

Of course, Alesker's result does not hold if the general linear group $GL(n)$ is replaced by the rotation group $SO(n)$.

Example: If K is a body of constant width, then all generalized $\mathcal{M}_{K,SO(n)}$-bodies are of constant width.

Results (joint work with Franz Schuster)

Theorem 1. *Let $B \in \mathcal{K}^n$ be non-symmetric. Then every neighborhood of B contains an affine image B' of B such that the set of generalized $\mathcal{M}_{B',SO(n)}$-bodies is dense in $\mathcal{K}^n$.*

Definition: $B \in \mathcal{K}^n$ is called universal if the expansion of h_B in spherical harmonics contains non-zero harmonics of all orders.

Theorem 2. *Let $B \in \mathcal{K}^n$. The set of generalized $\mathcal{M}_{B,SO(n)}$-bodies is dense in $\mathcal{K}^n$ if and only if B is universal.*

Theorem 3. *Let $B \in \mathcal{K}^n$ be non-symmetric. Then every neighborhood of B contains a universal affine image of B.*

Theorem 3 has a counterpart for symmetric bodies.

Basics on spherical harmonics

A spherical harmonic of order m on S^{n-1} is the restriction to S^{n-1} of a harmonic polynomial of order m on $\mathbb{R}^n$.

$\mathcal{H}_m^n$ vector space of spherical harmonics of order m

$N_{n,m}$ dimension of $\mathcal{H}_m^n$

$\mathcal{H}^n$ vector space of finite sums of spherical harmonics

Scalar product on $C(S^{n-1})$:

$$(f,g) := \int_{S^{n-1}} fg\,d\sigma$$

$Y_{m1}, \ldots, Y_{mN_{n,m}}$ a fixed orthonormal basis of $\mathcal{H}_m^n$

Let $\pi_m : C(S^{n-1}) \to \mathcal{H}_m^n$ denote the orthogonal projection, thus

$$\pi_m f := \sum_{j=1}^{N_{n,m}} (f, Y_{mj}) Y_{mj}, \qquad f \in C(S^{n-1}).$$

One calls

$$f \sim \sum_{m=0}^{\infty} \pi_m f$$

the condensed harmonic expansion of f.

Definition: $K \in \mathcal{K}^n$ is universal if $\pi_m h_K \neq 0$ for all $m \in \mathbb{N}_0$.

Remark: With $b(K)$ = mean width of K and $s(K)$ = Steiner point of K we have

$$(\pi_0 h_K)(u) = b(K)/2, \qquad (\pi_1 h_K)(u) = \langle s(K), u \rangle.$$

On the proof of Theorem 2

Theorem 2. *Let $B \in \mathcal{K}^n$. The set of generalized $\mathcal{M}_{B,SO(n)}$-bodies is dense in $\mathcal{K}^n$ if and only if B is universal.*

"$\Rightarrow$": If $\pi_m h_B = 0$ for some m, then $\pi_m h_K = 0$ for all generalized $\mathcal{M}_{B,SO(n)}$-bodies K and their limits. But there exists $M \in \mathcal{K}^n$ with $\pi_m h_M \neq 0$.

"$\Leftarrow$": Let B be universal.

Recall that for showing that a sufficiently smooth body K with center 0 is a generalized zonoid, we solved the integral equation

$$h_K(u) = \int_{S^{n-1}} |\langle u, v\rangle|\, f(v)\, d\sigma(v).$$

We try to solve a corresponding integral equation on the group $SO(n)$. Let ν be the normalized Haar measure on $SO(n)$.
Suppose we can solve the integral equation

$$h_K(u) = \int_{SO(n)} h_{\vartheta B}(u)\, f(\vartheta)\, d\nu(\vartheta).$$

Then we decompose $f = f^+ - f^-$ and get

$$h_K(u) + \underbrace{\int_{SO(n)} h_{\vartheta B}(u) f^-(v)\, d\nu(\vartheta)}_{h_{M_1}(u)} = \underbrace{\int_{SO(n)} h_{\vartheta B}(u) f^+(v)\, d\nu(\vartheta)}_{h_{M_2}(u)},$$

where $M_1, M_2 \in \mathcal{M}_{B,SO(n)}$ (approximate ν by discrete measures).

Since $K + M_1 = M_2$, the body K is a generalized $\mathcal{M}_{B,SO(n)}$-body.

It is sufficient to assume that $h_K \in \mathcal{H}^n$, because the set of such bodies is dense in $\mathcal{K}^n$.

How to solve

$$h_K(u) = \int_{SO(n)} h_{\vartheta B}(u)\, f(\vartheta)\, d\nu(\vartheta), \qquad u \in S^{n-1},$$

for

$$h_K = \sum_{m=0}^{k} \sum_{j=1}^{N_{n,m}} a_{mj} Y_{mj},$$

by a function f?

We use a kind of Fourier expansion on the group $SO(n)$.

The space $\mathcal{H}_m^n$ is invariant under the operation $(\vartheta f)(u) := f(\vartheta^{-1} u)$, where $\vartheta \in SO(n)$. Hence,

$$\vartheta Y_{mj}(u) = \sum_{i=1}^{N_{n,m}} t_{ij}^m(\vartheta) Y_{mi}(u), \qquad u \in S^{n-1},$$

with real coefficients $t_{ij}^m(\vartheta)$.

They satisfy orthogonality relations

$$N_{n,m} \int_{SO(n)} t_{ij}^{m} \, t_{sk}^{p} \, d\nu = \delta_{mp}\delta_{is}\delta_{jk}$$

and, as a consequence, for $f \in C(S^{n-1})$, the formula

$$\int_{SO(n)} \vartheta f(u) \, t_{ij}^{m}(\vartheta) \, d\nu(\vartheta) = N_{n,m}^{-1}(f, Y_{mj})Y_{mi}(u)$$

(it suffices to prove this for $f = Y_{kr}$).

Define $b_{mj} := (h_B, Y_{mj})$. Since B is universal, there is some j_m with $b_{mj_m} \neq 0$. Use this to define

$$f := N_{n,m} \sum_{m=0}^{k} \frac{1}{b_{mj_m}} \sum_{i=1}^{N_{n,m}} a_{mi} \, t_{ij_m}^{m}.$$

This function f solves the integral equation.

On the proof of Theorem 3 (the main result).

Theorem 3. *Let $B \in \mathcal{K}^n$ be non-symmetric. Then there exists $g \in GL(n)$, arbitrarily close to the identity, such that gB is universal.*

We explain the idea of the proof by demonstrating the 'easier half' of Theorem 3:

Proposition. *Let $B \in \mathcal{K}^n$ be non-trivial. Then there exists $g \in GL(n)$, arbitrarily close to the identity, such that $\pi_m h_{gB} \neq 0$ for all even numbers $m \in \mathbb{N}_0$.*

We reduce this to the fact that a segment S satisfies

$$\pi_m h_S \neq 0 \qquad \text{for all even } m.$$

(This is the reason for the solvability of the zonoid equation.)

In Cartesian coordinates, let Π_1 be the projection onto the x_1-axis, and suppose $\Pi_1 B =: S$ is a non-degenerate segment.

Define $g(\lambda) \in GL(n)$ by

$$g(\lambda) : (x_1, \ldots, x_n) \mapsto (x_1, \lambda x_2, \ldots, \lambda x_n).$$

For $\lambda \to 0$, the map $g(\lambda)$ converges to Π_1. It follows that

$$\lim_{\lambda \to 0} (h_{g(\lambda)B}, Y_{mj}) = (h_S, Y_{mj}).$$

If m is even, then $(h_S, Y_{mj_m}) \neq 0$ for some j_m.

Hence, the function

$$F(\lambda) := (h_{g(\lambda)B}, Y_{mj_m}), \qquad \lambda \in (0, 1],$$

does not vanish identically. This function is real analytic.

Hence, the set

$$Z_m := \{\lambda \in (0,1] : \pi_m h_{g(\lambda)K} = 0\}$$

is countable. This holds for each even m.

Therefore, every neighborhood of 1 contains some λ with

$$\pi_m h_{g(\lambda)K} \neq 0 \qquad \text{for all even } m.$$

This completes the proof of the Proposition.

Strategy for the 'second half', i.e., K non-symmetric, m arbitrary

Recall the claim:

Theorem 3. *Let $B \in \mathcal{K}^n$ be non-symmetric. Then there exists $g \in GL(n)$, arbitrarily close to the identity, such $\pi_m h_{gB} \neq 0$ for all m.*

1.) Prove the two-dimensional case of Theorem 3.

2.) Lemma. *If $B \subset \mathbb{R}^2 \subset \mathbb{R}^n$ and B is universal in $\mathbb{R}^2$, then B is universal in $\mathbb{R}^n$.*

3.) Similarly as before, use linear maps converging to the projection onto $\mathbb{R}^2$.

We indicate only Step 1.

The two-dimensional case

Let $B \subset \mathbb{R}^2$ be a non-symmetric convex body.

Write $h_B((\cos\varphi, \sin\varphi)) =: h_B(\varphi)$.

The space $\mathcal{H}_m^2$ is spanned by the functions $\cos m\varphi$ and $\sin m\varphi$. Therefore, in complex notation

$$\pi_m h_{gB} = 0 \iff \int_0^{2\pi} h_{gB}(\varphi) \mathrm{e}^{\mathrm{i}m\varphi} d\varphi = 0.$$

Define a map $F_{B,m} : GL(2)^+ \to \mathbb{C}$ by

$$F_{B,m}(g) := \int_0^{2\pi} h_{gB}(\varphi) \mathrm{e}^{\mathrm{i}m\varphi} d\varphi \qquad \text{for } g \in GL(2)^+.$$

This map is real analytic.

Proposition. *The relation*

$$F_{B,m}(g) = \int_0^{2\pi} h_{gB}(\varphi)\mathrm{e}^{\mathrm{i}m\varphi}d\varphi = 0 \qquad \textit{for all } g \in GL(2)^+$$

cannot hold for any odd integer $m \geq 1$.

For the proof, let m be a smallest counterexample. We use

$$g(\lambda) \sim \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \qquad \text{and} \qquad R(\alpha) \sim \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}.$$

Consider the first map. From $h_{g(\lambda)B}(\varphi) = \sqrt{\cos^2\varphi + \lambda^2\sin^2\varphi}\, h_B(\psi)$ and a substitution we get

$$F_{B,m}(g(\lambda)) = \lambda^2 \int_0^{2\pi} h_B(\psi) \frac{(\lambda\cos\psi + \mathrm{i}\sin\psi)^m}{(\lambda^2\cos^2\psi + \sin^2\psi)^{\frac{m+3}{2}}}\, d\psi.$$

Since this vanishes for all $\lambda \in (0,1]$, the derivative with respect to λ at 1 vanishes. This yields

$$\int_0^{2\pi} h_B(\psi)[(3-m)\,\mathrm{e}^{\mathrm{i}(m-2)\psi} + (3+m)\,\mathrm{e}^{\mathrm{i}(m+2)\psi}]\,d\psi = 0.$$

Now we use the second map. Since $F_{B,m}(R(\alpha)) = 0$ for α in a neighborhood of 0, the preceding holds with $\psi + \alpha$ instead of ψ in the exponents. This yields

$$\begin{aligned} \int_0^{2\pi} h_B(\psi)\,\mathrm{e}^{\mathrm{i}(m-2)\psi}\,d\psi &= 0 \quad \text{for } m \neq 3, \\ \int_0^{2\pi} h_B(\psi)\,\mathrm{e}^{\mathrm{i}(m+2)\psi}\,d\psi &= 0. \end{aligned}$$

Now the existence of a smallest counterexample m leads to a contradiction.