AIM Workshop
COHOMOLOGY AND REPRESENTATION THEORY FOR FINITE GROUPS OF LIE TYPE: COMPUTATIONS AND CONSEQUENCES

Organized by Chris Bendel, Terrell Hodge, Cornelius Pillen, and Brian Parshall

PROBLEM AND DISCUSSION LIST

Annotated
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This document records some issues, problems, and discussion from the AIM Workshop of June 25–29, 2007. The original core was a list of questions and issues to be pursued, culled from discussions in the first two days of the workshop. Originally constituted as a "list of open problems", some of the questions as posed were for the benefit of group members seeking clarity regarding what was known about solutions. Others were truly regarded as open problems in the usual language of mathematical researchers; among these, factors such as their importance for the field, potential accessibility, and/or perceived applicability of computational approaches varied widely.

The purpose of this draft is to collect outcomes of discussions in an arrangement similar to the last draft circulated at the workshop, for the benefit of workshop participants. Subsequently, the organizers will cull a final list of "open problems" for future investigation by researchers.

I. Kazhdan-Lusztig (KL) polynomials for algebraic groups, character formulas, filtrations and Ext-groups.

NOTE: Z. Lin's AIM workshop talk provides some background on this topic and will be referred to later below; see www.aimath.org/WWN/finiteliegps.

1. How to compute KL (Kahzdan-Lusztig) polynomials, what has been done, and what is available? **Add links, references.**

   (a) GAP, CHEVIE (can deal with the affine Weyl groups), in small cases. **Add examples; see also samples posted at www.aimath.org/WWN/finiteliegps.**

   (b) Programs of Fokko du Cloux.

   Status: During the workshop, the group learned a bit about this from A. Noël, and F. Lübeck; of particular interest (see item * below) is the apparent ability of du Cloux's programs for calculating affine Kazhdan-Lusztig polynomials rather than just KL polys for finite Weyl groups which were all that were needed for the ATLAS calculations for real Lie groups (e.g., for type E_8). Noël and Lübeck held further discussions about some program details and Lübeck noted the potential and interest in creating some direct interface between some ATLAS programs and GAP, to increase accessibility and lend other advantages.

   (c) L. Scott has done work with undergraduate students on computing KL polynomials. For examaple, a list of KL polynomials for $SL(6, \overline{\mathbb{F}}_7)$ can be found on his web page: http://www.math.virginia.edu/~lls2l/research_undergrad.html.

Parabolic KL polynomials W_a/W (for W_a w.r.t. W) are the "essential" KL polynomials for modular cases.

2. What is the relation between radical filtrations (e.g., as in Z. Lin's talk) and KL polynomials? Does the radical filtration determine the KL polynomials? What if you have knowledge of the extensions between standard modules and irreducible modules (or between irreducible modules and co-standard modules)?

   Status: See [CPS1]. See [Irving].

3. Representations with singular weights instead of regular weights.

   Discussion: (parity condition). Write down $\mathrm{Ext}(\Delta(\lambda), L(\mu))$ polynomials.

The original Lusztig conjecture was simply a character formula. The version involving dimensions of Ext groups stated by Z. Lin in his AIM lecture is a later formulation. The original statement of the conjecture involved KL polynomials evaluated at 1.

4. Lin's conjecture; see [Lin1], also [Lin2]. For $F(\lambda)$ the interior of a facet,

$$[\mathrm{Rad}^i \,/\, \mathrm{Rad}^{i+1}\, T_\lambda^\mu M_{y\cdot\lambda} : T_\lambda^\mu L_{w\cdot\lambda}] = [\mathrm{Rad}^i \,/\, \mathrm{Rad}^{i+1}\, M_{y\cdot\lambda} : L_{w\cdot\lambda}]$$

   where $y \cdot \mu \in \hat{F}(y \cdot \lambda)$ and $w \cdot \mu \in \hat{F}(w \cdot \lambda)$ lie in the upper closure, with $T_\lambda^\mu M_{y\cdot\lambda} = M_{y\cdot\mu}$ and $T_\lambda^\mu L_{w\cdot\lambda} = L_{w\cdot\mu}$. The conjecture has been verified for types A_2, B_2 in the Jantzen region.

   Status: This problem seems to be "hard".

5. Can we compute the radical filtrations of the Weyl modules, and what tools are there for making the computations? Same question for the Jantzen filtration (eventually). Calculations of the Jantzen filtration have been done for "small" dimensional cases.

6. Radical length of the Weyl module.

   Status: See the item immmediately above.

7. Connections between KL polynomials and quivers /relations. (By this, the group meant to get to information about basic algebras associated to representation theory.)

   Status: The group did not get to revisit this question.

8. Lusztig conjectures for non-restricted representations of Lie algebras in positive characteristic. (Can you get the relations?)

   Status: The group did not get to revisit this question.

9. Compute Kazhdan-Lusztig polynomials for small rank groups. Potentially verify or find a counter-example to the Lusztig conjecture for algebraic groups in prime characteristic.

   Status: Brian B. crossed out first problem statement from group discussion – get specific references/links/reasons. For the second, at the start of the workshop, Bobbe Cooper thought she might have a counterexample to the Lusztig conjecture; working with Dan Rozemond and Frank Lübeck during the conference, she was able to compute the data she needed to determine that she did not have a counterexample. However, Brian Boe wrote that Boe, Cooper, and Lübeck will work further on this for cases $SL_5(11)$, $SL_6(7)$.

10. Ext groups between Weyl modules and irreducible modules for very big weights for SL_2.

    Status: Not easy–beyond the Jantzen region. See [CPS2].

11. What is the [LLT] approach to KL polynomials?

    Status: Look up/ask them? Parabolic H. S.?

12. What happens with standardly-stratified algebras?

    Status: Not discussed by the group; was suggested by P. Webb, so seek further input from him, also L. Scott.

II. Cross-characteristic Representation Theory

A survey on this topic appears in B. Srinivasan's AIM workshop lecture at www.aimath.org/WWN/finiteliegps/.

1. How do KL polynomials enter into the ℓ-decomposition numbers for $GL_n(q)$?

   Status: This is Dipper-James theory; answer is known by results of Cline-Parshall-Scott.

2. In a cross-characteristic representation, what is the smallest field in which it can be realized? I.e., determine the splitting fields for the finite groups of Lie type.

   Discussion and status: Asked by Á. Seress. Group viewed it as interesting question. B. Srinivasan didn't know the answer.

3. Analogous highest-weight categories, double affine Hecke algebras, complex reflection groups, Cherednik algebras. Some of the above questions in the more general context. (Char. 0 stuff.)

   Discussion and status: The Cherednick algebras give analogs of $q-$Schur algebras for all finite Weyl groups. (See [RGO] for category $\mathcal{O}$ for Cherednick algebras in the type A case.) Are there similar analogs for all complex reflection groups? Can they be used to understand the cross-characteristic representations for finite groups of Lie type a la Dipper and James?

4. Compute the cohomology ring of finite groups of Lie type in non-describing characteristic. Compare Quillen's work for $GL_n(q)$.

   Discussion and status: Suggest checking work of Fedorowicz and Priddy (Springer Lecture Notes). B. Parshall suggests problem is doable; can look at Quillen's paper. Answer could even be well known.

5. Find a "James Conjecture" types other than type A (even in the defining characteristic). See [James1] for a reference.

   Discussion and status:

III. Symmetric Groups

NOTE: For background on many of the topics below, see D. Hemmer's AIM workshop talk at www.aimath.org/WWN/finiteliegps/.

1. In work of J. Carlson, N. Mazza, and D. Nakano [CMN] on endo-trivial modules, the existence of a certain endo-trivial module for Σ_{2p} is shown but without an explicit description. Find a module-theoretic description of this module.

   Status: This problem is expected to be doable. Classifying endo-trivial modules for Σ_d, $d \geq p^2$ is a related problem.

2. On Ext for symmetric groups:

   - Compute $\mathrm{Ext}^1_{\Sigma_{15}}(k, S^{(5,5,5)}) = \mathrm{H}^1(\Sigma_{15}, S^{(5,5,5)})$.
   - More generally, investigate $\mathrm{Ext}^i_{\Sigma_{p^a n}}(S^{p^a\lambda}, S^{p^a\mu})$ for fixed p, λ, μ as a increases with the goal of finding computational evidence that this stabilizes. (e.g., analogous to the generic cohomology of [CPSvdK]).

   Status: The first item above is considered doable; the significance is that for $p = 5$ and it is the only outstanding case (in calculations made by Dave Hemmer). The difficulty of the second problem isn't clear; for the p-restricted case, the results are already known.

3. When does a Specht module have simple socle?

   Status: Hard! More generally, one wants to compute the socle of a Specht module.

4. Given extensions between simple modules D_λ, D_μ for S_n, do λ, μ have to be comparable?

   Status: In the large, this is a hard problem. However, one could try here to compute some small examples.

5. For Type A, the James' Conjecture implies a conjecture for the Hecke algebra of a symmetric group. Make calculations to confirm this.

   Status: This should be doable.

6. Compute the basic algebra for the Schur algebra $S(3,r)$ (or more generally, $S(n,r)$) over fields of (small) prime characteristic. Also check Koszulity of Schur algebras.

IV. Support Varieties

NOTE: For background on this topic, see for example [B], [C], or [N].

1. Compute examples of support varieties of Weyl modules for higher Frobenius kernels G_r, $r > 1$.

   Status: Done for $G = SL_2$ [SFB]. It looks potentially doable for $G = SL_3$ and $r = 2$. For higher r and higher rank groups, the structure of $C_r(N_1(\mathfrak{g}))$ is complicated. Indeed, a better understanding of $C_r(N_1(\mathfrak{g}))$ (the variety of pairwise-commuting r-tuples of $[p]$-nilpotent elements of $\mathfrak{g}$) is crucial. A related implementation challenge which would be useful for computation would be to implement the group algebra $k[G_r]^*$ in a CAS.

2. The AIM Conjecture. Let G be a simple algebraic group over an algebraically closed field of characteristic p. Assume $p \geq h$. Then $V_{G_r}(H^0(\lambda)) = V_{G_r}$ if and only if λ is a p-regular weight.

   Status: The preceding computations would give evidence for/against this conjecture. Seems like its confirmation or rejection should be doable.

3. On maximal elementary abelian subgroups of $GL_n(p)$.

   - Determine the smallest rank of a maximal elementary abelian subgroup.
   - Estimate the number of conjugacy classes of maximal elementary abelian subgroups.
   - More precisely, attempt to answer up to say $GL_{20}(p)$ in hopes of seeing a general pattern.
   - Larger goal: Understanding the Quillen category and support varieties.

   Status: On the first part, we know that the answer lies between $n/2$ and $n - 1$. Motivated by small group problem sessions early on, during the workshop, Á. Seress implemented a program to construct elementary abelian subgroups. This is a start towards some further experimental results for small n. The second problem seems hard but with a good computer program, might be able to get some results for small ranks. On part (3), $n = 20$ may likely be beyond current computer technology. But smaller ranks might be sufficient to see some patterns. In general, particularly the larger goal, is a fundamental problem, which while hard, seems reasonable for further investigation.

4. Compute the support varieties of Weyl modules for $SL_2(p^r)$ ($r \geq 2$). Probably can be done. Then do $SL_3(p^r)$ and higher, more generally, ...

   Status: Seems doable for SL_2. And maybe SL_3. Higher??

5. Further connections between support varieties of G_r and $G(\mathbb{F}_{p^r})$.

   Status: For $r = 1$, there are nice connections in [CLN]. This is definitely a problem worthy of further study.

V. Other Theoretical Questions.

1. In almost simple finite groups of Lie type, what are the smallest and second-smallest nontrivial conjugacy classes?

   Status: Done in many/most cases and should generally be doable. References?

2. When does one simple group sit inside another?

   Status: This is a "lifetime" problem which people are working on. E.g., Aschbacher, Lübeck-Seitz.

3. When does an irreducible representation of a simple group restrict to an irreducible representation of a simple subgroup?

   Status: Another lifetime problem which people are working on. E.g., Saxl.

4. Given an irreducible representation of a finite group, how do you tell which classical groups it lies in?

   Status: This is another hard problem.

5. The "categorification" process.

   Status: Short overviews of this were provided at the workshop by P. Webb and B. Srinivasan. For more information see [K], ??

6. On Auslander-Reiten Quivers:

   - Compute the relative A-R quiver for modules over a Schur algebra (in small cases) having a Δ-filtration.
   - For a finite group, say $C_2 \times C_4$ over a field of characteristic 2 to start, consider modules M, N in the identity component of the A-R quiver. Determine whether the non-projective indecomposable factors of $M \otimes N$ also lie in the identity component.

   Status: The problems were only briefly discussed. They seem interesting and potentially doable at least in small cases.

7. Determine whether a projective module for a Schur algebra is self-dual. There exists a complicated conjecture of Donkin-De Visscher on when this should be true. Try to verify/improve conjecture in small cases.

   Status: Motivated by discussions early in the workshop, during the workshop, J. Carlson modified some previous programs to calculate information on Schur algebras; thus one can now see some such computations posted at http:www.math.uga.edu/ jfc/schur.html. These examples should be able to provide some further information giving evidence for Koszulity of the Schur algebra. Get further updates from B. Parshall.

8. Is the Ext-algebra over $\mathbb{C}$ for a Hecke algebra at a root of unity for a finite Coxeter group finitely generated?

   Status: Likely hard as has been the case with most other finite generation problems.

9. Comparing finite group cohomology and algebraic group cohomology in the sense of the work of [CPSvdK]. E.g, can we get more precise information on when they agree, particular bounds that do not depend on the module?

   Status: New work of [CPS2] in conjunction with somewhat recent work of [BNP] should provide some better information here. In particular, there are long-standing conjectures of Guralnick on the dimensions of cohomology groups of finite groups. Combining these recent results will give some evidence toward the validity of these conjectures.

VI. Computer Implementation Wish List

1. Specht modules for symmetric groups

   Status: Doable in GAP.

2. Weyl modules for algebraic groups in prime characteristic

   Status: Doable in GAP but not easily accessible at the moment, i.e., not implemented in a public form. People can obtain specific cases by contacting F. Lübeck, who will run the programs he has designed.

3. Group algebras for Frobenius kernels, i.e., $kG_r = k[G_r]^*$. Perhaps inside a product of matrix algebras.

   Status: A good problem and potentially doable.

4. More generally, given a finite-dimensional Hopf algebra A implemented in a CAS, can the dual A^* be computed and presented in a useful manner?

   Status: Feasibility was not discussed.

5. More efficient computations of centralizers.

   Status: This would be helpful for the maximal elementary abelian subgroup problem mentioned in Part IV.

6. Computational work done on basic algebras for algebras associated with module categories of bounded highest weight for algebraic groups in characteristic p.

   Status: ??

7. How might you try to compute the cohomology of finite groups? What are the current size limitations?

   Status: J. Carlson has Magma programs for computing cohomology rings in terms of generators and relations (for not too large groups) [CTVZ]. There are also some algorithms implemented in GAP for computing cohomology groups in small degrees.

8. Calculations of quantum groups at p^n-roots of unity, $n > 1$.

   Status: Can be done using the "Quagroup" package in GAP. Are more tools needed than what are there?

9. What are the "nilpotent" elements in small quantum groups? How do you go about finding them? Same question for $u_q(sl_3)$.

   Status: One should be able to use the "Quagroup" package to do some experimental work on this question.

10. Methods for presenting computational data (especially on the internet). Modifying existing programs to make data more presentable. Obtaining assistance in posting to the web data/programs one already has. (Is there funding for this?)

    Status: This was discussed somewhat. It remains a fundamental problem.

VII. Computer Related Activities For The Workshop

1. Overview of where computing is right now. Why would you use one system over another, or why it wouldn't matter. What algorithms are there/aren't there? In what areas should we invest computational efforts? What are our computational needs for the future (in a general sense)?

2. The real Lie groups project.

3. 5-10 minute lecture demo on wikis.

4. Demonstrations of packages for GAP, Chevie, Lie, e.g. `Quagroup` package for GAP.

Status: These activities were all accomplished (at least to some extent) during the workshop. Related materials, including:

- Lectures by J. Carlson and F. Lübeck on computational methods and applications
- Lecture by A. Noël on the ATLAS of Lie Groups project and their computation of KL polynomials
- GAP and CHEVIE demonstrations by F. Lübeck and C. Bendel
- Magma and LIE demonstrations by D. Rozemond
- Information about the wiki page set up by T. Hodge for resources for conference topics
- Links and information for accessing and/or downloading GAP and Magma

can be found at www.aimath.org/WWN/finiteliegps/. Informal discussions suggest the hands-on portions and demonstrations, with time for lots of questions, were particularly valued, setting the stage for further use of technology by those who had little to no experience with it and leading to fruitful discussions between those with technical expertise and those with computational needs, thus achieving one of the main goals of the workshop. The issue of providing well (say, in some integrated form, rather than by the initiative or generosity of a few people) for current and future computational needs remains fairly open and unresolved.

VII. Closing Reflections by Workshop Participants.

1. Interest level in running another workshop on this topic (integrating computation and research and acquiring/maintaining/enhancing computational proficiency), and/or with other related theoretical topics as basis.

2. General interest in AIM workshop format, and in similarities with UGA VIGRE activities (workshop reports by latter group resulted in a lot of lively questions and discussion). Discussions of virtue of such formats, and of constructing other educational activities for faculty and for students this way.

3. Discussion of further funding opportunities, e.g., SQUARES at AIM (but not 'repeat' workshops), AIM assistance with seeking NSF FRG grants and/or ability of AIM to serve as conference/workshop site w/administrative (although not financial) support.

4. Discussions regarding collecting and promoting results of workshop and furthering access to computational methods via webpages: AIM and wiki sites. Maintaining, linking, accessing them and creating a consortium focused on needs for modular representation theorists.

5. Missing items:

   - Did not take have, as concretely as would have liked, 'take- away' problems for working with undergrads, or formalized collaborations for big-group projects.
   - Some frustration with lack of good computing facilities at AIM and problems with internet access at Creekside Inn.

6. Special kudo: AIM wine selection best of any conference ever.

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