Semidefinite Programming Rank Reduction for Graph Realization and Sensor Network Localization

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Outline

- Semidefinite Programming (SDP)
- SDP Rank Theorems
- Graph Realization and Sensor Network Localization
- Universal Rigidity and SDP Rank
- More Questions?

Semidefinite Programming Problem

Consider the Semidefinite Programming problem:

$$
\begin{array}{llll}
(SDP) & \text{minimize} & A_0 \bullet X & \\
 & \text{subject to} & A_i \bullet X = b_i & i = 1, \ldots, m, \\
 & & X \succeq \mathbf{0} &
\end{array}
$$

where $A_0, A_1, \ldots, A_m$ are given $n \times n$ symmetric matrices and $b_1, \ldots, b_m$ are given scalars, and

$$A \bullet X = \sum_{i,j} a_{ij} x_{ij} = \text{trace}(A^T X).$$

The Dual of SDP

The dual problem to (SDP) can be written as:

$$
\begin{array}{lll}
(SDD) & \text{maximize} & \mathbf{b}^T\mathbf{y} \\
 & \text{subject to} & \sum_i^m y_i A_i + S = A_0,\ S \succeq \mathbf{0},
\end{array}
$$

where $\mathbf{y} = (y_1; \ldots; y_m) \in \mathcal{R}^m$.

Let X^* and S^* be a solution pair with zero duality gap. Then

$$\text{rank}(X^*) + \text{rank}(S^*) \leq n.$$

Thus, if there is S^* such that $\text{rank}(S^*) \geq n - d$, then the max rank of X^* is bounded above by d.

Computational Complexity and Rank of SDP Solution

- The SDP interior-point algorithm finds an ϵ-approximate solution where solution time is linear in $\log(1/\epsilon)$ and polynomial in m and n.
- Barvinok 95 (earlier results ?)showed that if the problem is solvable, then there exists a solution X^* whose rank r satisfies $r(r+1) \leq 2m$. (A constructive proof can be based on Carathéodory's theorem.)
- And the rank bound is essentially tight.
- A such optimal solution can be found in polynomial time; Pataki (1999), and Alfakih/Wolkowicz (1999).

SDP Feasibility Problem

For simplicity, consider finding X satisfies

$$A_i \bullet X = b_i \qquad i = 1, \ldots, m, \quad X \succeq \mathbf{0}$$

where $A_1, \ldots, A_m$ are positive semidefinite matrices and scalars $(b_1, \ldots, b_m) \geq \mathbf{0}$.

$$x_1 + x_2 + x_3 = 1,$$
$$\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \succeq \mathbf{0}.$$

What is the rank of SDP solution matrices?

Low-Rank SDP Solution

- We are interested in finding a fixed low–rank (say d) solution to the above system.
- However, there are some issues:
  - Such a solution may not exist!
  - Even if it does, one may not be able to find it efficiently.
- So we consider an approximation of the problem.

Approximate Low-Rank SDP Solution

We consider the problem of finding an $\hat{X} \succeq 0$ of rank at most d that satisfies every SDP constraint approximately and uniformly:

$$\beta(m, n, d) \cdot b_i \leq A_i \bullet \hat{X} \leq \alpha(m, n, d) \cdot b_i \quad \forall\ i = 1, \ldots, m.$$

Here, $\alpha(\cdot) \geq 1$ and $\beta(\cdot) \in (0, 1]$ are called the distortion factors. Clearly, the closer are both to 1, the better the solution quality.

Approximate Low-Rank Theorem (So, Y and Zhang 07)

Let $r = \max\{\text{rank}(A_i)\}$. Then, for any $d \geq 1$, there exists an $\hat{X} \succeq \mathbf{0}$ with $\text{rank}(\hat{X}) \leq d$ such that

$$\alpha(m,n,d) = \begin{cases} 1 + \dfrac{12\ln(4mr)}{d} & \text{for } 1 \leq d \leq 12\ln(4mr) \\ 1 + \sqrt{\dfrac{12\ln(4mr)}{d}} & \text{for } d > 12\ln(4mr) \end{cases}$$

$$\beta(m,n,d) = \begin{cases} \dfrac{1}{e(2m)^{2/d}} & \text{for } 1 \leq d \leq 4\ln(2m) \\ \max\left\{\dfrac{1}{e(2m)^{2/d}},\ 1 - \sqrt{\dfrac{4\ln(2m)}{d}}\right\} & \text{for } d > 4\ln(2m) \end{cases}$$

Moreover, there exists an efficient randomized algorithm for finding such an $\hat{X}$.

Some Remarks

- There is always a low-rank approximate SDP solution with bounded distortion factors.
- As the allowable rank increases, the distortion become smaller and smaller. In particular, when $d = O(\ln(m))$, the distortion factors are both equal a constant close to 1.
- The lower distortion factor is independent of n and the rank of A_is.
- The factors are sharp; but they can be improved if we only consider one–sided inequalities.
- This result contains as special cases several well-known results in the literature.

Low Rank SDP Applications

The low–rank SDP problem arises in many applications, e.g.:

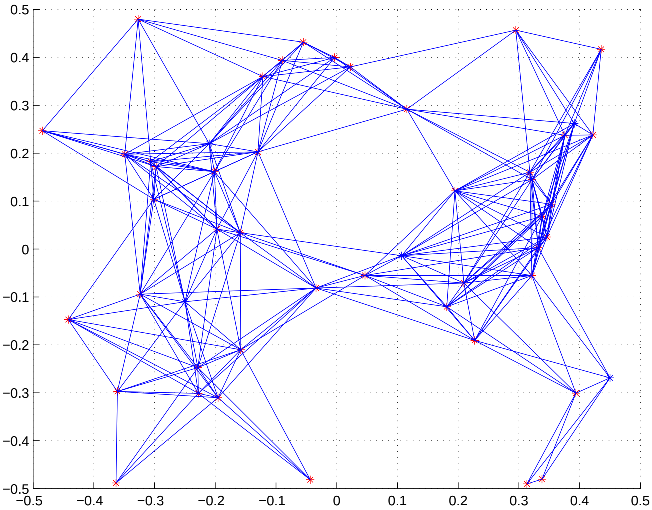
- metric embedding/dimension reduction (e.g., Johnson and Lindenstrauss 84, Matousek 90, Sun, Xiao and Boyd 06, etc.)
- approximating non-convex (real, complex) quadratic optimization (e.g., Goemans and Williamson 95, Nesterov 98, Y 98, Nemirovskii, Roos and Terlaky 99, Luo, Sidiropoulos, Tseng and Zhang 06, So, Zhang and Y 07, etc.)
- distance matrix completion (e.g., Laurent 97, Alfakih, Khandani and Wolkowicz 99, etc.)
- low-rank matrix completion (e.g., ISMP 2009 ...)
- graph realization/sensor network localization (e.g., Biswas and Y 04, So and Y 04, Biswas, Toh, and Y 06, Jin and Saunders 07, Wang, Zheng, Boyd and Y 08, Kim, Kojima and Waki 08, Pong and Tseng 08, Krislock and Wolkowicz 08, etc.)

Graph Realization and Sensor Network Localization

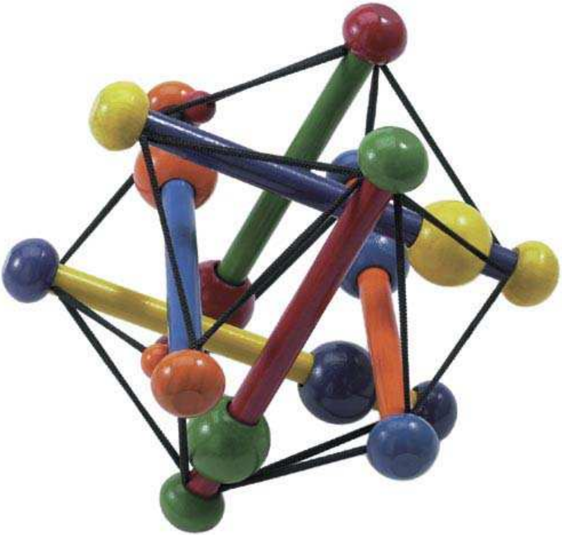
Given a graph $G = (V, E)$ and sets of non–negative weights, say $\{d_{ij} : (i,j) \in E\}$, the goal is to compute a realization of G in the Euclidean space $\mathbf{R}^d$ for a given low dimension d, i.e.

- to place the vertices of G in $\mathbf{R}^d$ such that
- the Euclidean distance between a pair of adjacent vertices (i,j) equals to (or bounded by) the prescribed weight $d_{ij} \in E$.

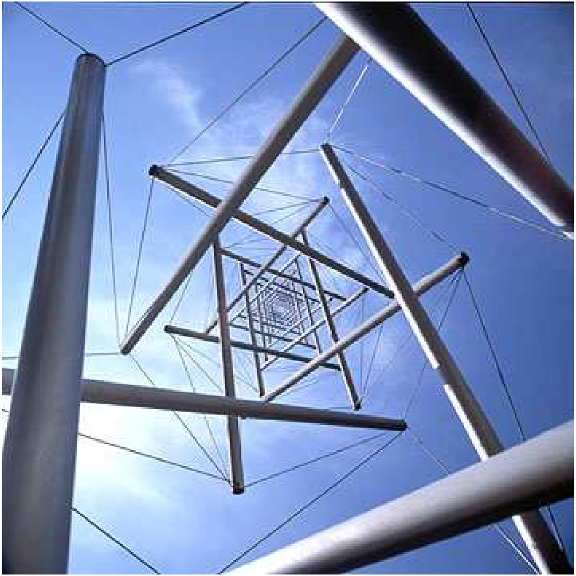
Unit-Disk Sensor Network: 50-node in 2-D



Tensegrity Graph: a Toy Graph Realization



Tensegrity Graph: a Needle Tower Realization



Molecular Conformation

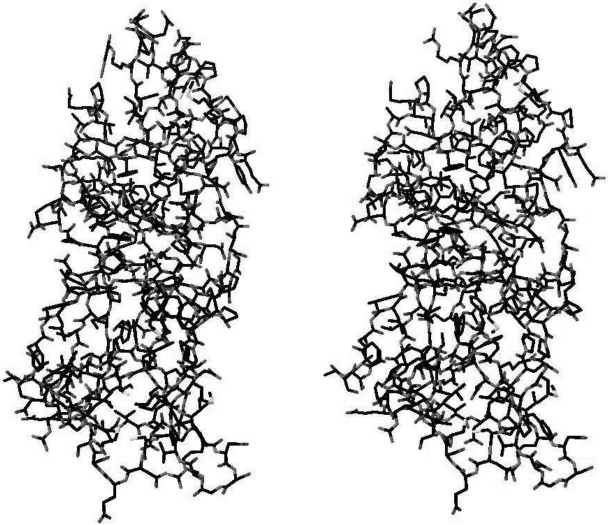


Figure: 1F39 with 85% of distances below 6Å and 10% noise

Sensor Localization Problem

Given anchors $\mathbf{a}_k \in \mathbf{R}^d$, $\hat{d}_{kj} \in N_a$ and $d_{ij} \in N_x$, find $\mathbf{x}_i \in \mathbf{R}^d$ such that

$$\begin{aligned} \|\mathbf{x}_i - \mathbf{x}_j\|^2 &= d_{ij}^2, \ \forall\ (i,j) \in N_x,\ i < j, \\ \|\mathbf{a}_k - \mathbf{x}_j\|^2 &= \hat{d}_{kj}^2, \ \forall\ (k,j) \in N_a, \end{aligned}$$

(ij) ((kj)) connects points $\mathbf{x}_i$ and $\mathbf{x}_j$ ($\mathbf{a}_k$ and $\mathbf{x}_j$) with an edge whose Euclidean length is d_{ij} ($\hat{d}_{kj}$).

Does the system have a localization or realization of all $\mathbf{x}_j$'s? Is the localization unique? Is there a certification for the solution to make it reliable or trustworthy? Is the system partially localizable with certification? All these questions are related to Global Optimization.

For simplicity, we fix $d = 2$ in the following.

Matrix Representation I

Let $X = [\mathbf{x}_1\ \mathbf{x}_2\ \ldots\ \mathbf{x}_n]$ be the $2 \times n$ matrix that needs to be determined and $\mathbf{e}_j$ be the vector of all zero except 1 at the jth position. Then

$$\mathbf{x}_i - \mathbf{x}_j = X(\mathbf{e}_i - \mathbf{e}_j) \quad \text{and} \quad \mathbf{a}_k - \mathbf{x}_j = [I \quad X](\mathbf{a}_k; -\mathbf{e}_j)$$

so that

$$\|\mathbf{x}_i - \mathbf{x}_j\|^2 = (\mathbf{e}_i - \mathbf{e}_j)^T X^T X(\mathbf{e}_i - \mathbf{e}_j)$$

$$\|\mathbf{a}_k - \mathbf{x}_j\|^2 = (\mathbf{a}_k; -\mathbf{e}_j)^T [I \quad X]^T [I \quad X](\mathbf{a}_k; -\mathbf{e}_j) =$$

$$(\mathbf{a}_k; -\mathbf{e}_j)^T \begin{pmatrix} I & X \\ X^T & X^T X \end{pmatrix} (\mathbf{a}_k; -\mathbf{e}_j).$$

Matrix Representation II

Or, equivalently,

$$
\begin{aligned}
&(\mathbf{e}_i - \mathbf{e}_j)^T Y (\mathbf{e}_i - \mathbf{e}_j) = d_{ij}^2, \ \forall\ i, j \in N_x,\ i < j, \\
&(\mathbf{a}_k; -\mathbf{e}_j)^T \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} (\mathbf{a}_k; -\mathbf{e}_j) = \hat{d}_{kj}^2, \ \forall\ k, j \in N_a, \\
&Y = X^T X.
\end{aligned}
$$

SDP Relaxation

Change

$$Y = X^T X$$

to

$$Y \succeq X^T X.$$

This matrix inequality is equivalent to

$$\begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} \succeq \mathbf{0},$$

This matrix has rank at least 2. If it's 2, then $Y = X^T X$, and the converse is also true.

SDP Relaxation in Standard Form

$$Z = \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix}.$$

Find a symmetric matrix $Z \in \mathbf{R}^{(2+n)\times(2+n)}$ such that

$$\begin{array}{l} Z_{1:2,1:2} = I \\ (\mathbf{0}; \mathbf{e}_i - \mathbf{e}_j)(\mathbf{0}; \mathbf{e}_i - \mathbf{e}_j)^T \bullet Z = d_{ij}^2, \ \forall\ i,j \in N_x,\ i < j, \\ (\mathbf{a}_k; -\mathbf{e}_j)(\mathbf{a}_k; -\mathbf{e}_j)^T \bullet Z = \hat{d}_{kj}^2, \ \forall\ k,j \in N_a, \\ Z \succeq \mathbf{0}. \end{array}$$

If every sensor point is connected, directly or indirectly, to an anchor point, then the solution set must be bounded.

The Dual of the SDP Relaxation

$$\begin{array}{ll} \text{minimize} & I \bullet V + \sum_{i<j\in N_x} w_{ij} d_{ij}^2 + \sum_{k,j\in N_a} \hat{w}_{kj} \hat{d}_{kj}^2 \\ \text{subject to} & \begin{pmatrix} V & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + \sum_{i<j\in N_x} w_{ij} (\mathbf{0}; \mathbf{e}_i - \mathbf{e}_j)(\mathbf{0}; \mathbf{e}_i - \mathbf{e}_j)^T \\ & + \sum_{k,j\in N_a} w_{kj} (\mathbf{a}_k; -\mathbf{e}_j)(\mathbf{a}_k; -\mathbf{e}_j)^T \succeq 0, \end{array}$$

where variable matrix $V \in \mathcal{M}^2$, variable w_{ij} is the (stress) weight on edge between $\mathbf{x}_i$ and $\mathbf{x}_j$, and $\hat{w}_{kj}$ is the (stress) weight on edge between $\mathbf{a}_k$ and $\mathbf{x}_j$.

Note that the dual is always feasible since $V = \mathbf{0}$ and all $w.$ equal 0 is a feasible solution.

The rank of any optimal dual slack matrix is less or equal to n.

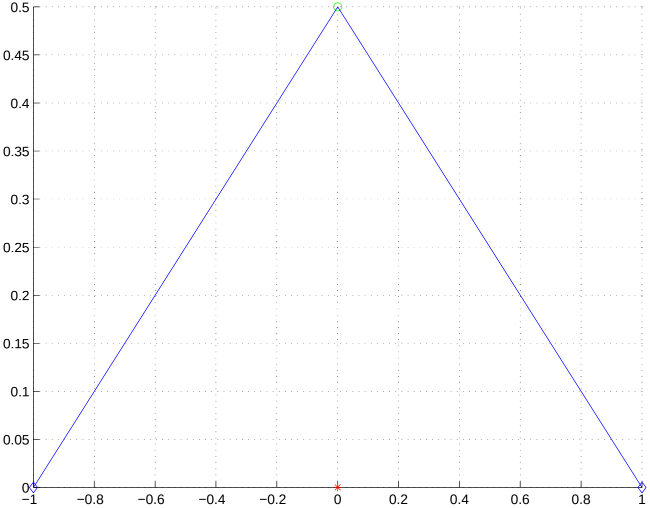
Unique Localizability

A sensor network is 2-uniquely-localizable if there is a unique localization in $\mathbf{R}^2$ and there is no $\mathbf{x}_j \in \mathbf{R}^h,\ j = 1, ..., n$, where $h > 2$, such that

$$\begin{aligned} \|\mathbf{x}_i - \mathbf{x}_j\|^2 &= d_{ij}^2,\ \forall\ i,j \in N_x,\ i < j, \\ \|(\mathbf{a}_k; \mathbf{0}) - \mathbf{x}_j\|^2 &= \hat{d}_{kj}^2,\ \forall\ k,j \in N_a. \end{aligned}$$

The latter says that the problem cannot be localized in a higher dimension space where anchor points are simply augmented to $(\mathbf{a}_k; \mathbf{0}) \in \mathbf{R}^h$, $k = 1, ..., m$.

One sensor-Two anchors: Not localizable



Uniquely-Localizable Graphs

- If every edge length is specified, then the sensor network is 2-uniquely-localizable (Schoenberg 1942).
- If one sensor with its edge lengths to at least three anchors (in general positions) specified, then it is 2-uniquely-localizable (So and Y 2005).

ULPs can be localized by SDP

Theorem

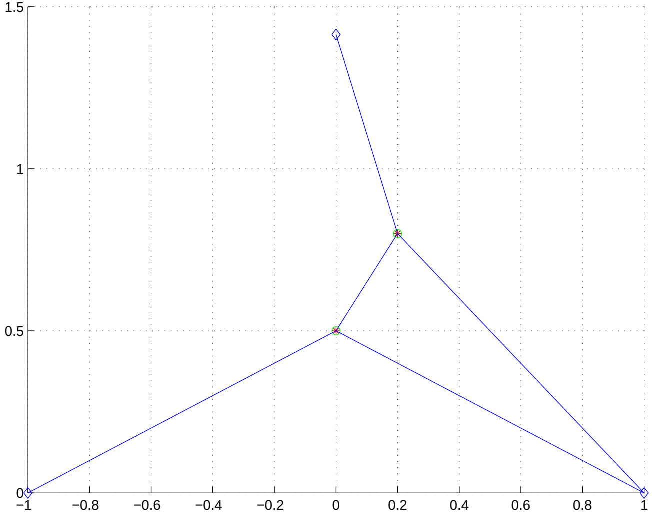
(So and Y 2005) The following statements are equivalent:

1. *The sensor network is 2-uniquely-localizable;*
2. *The max-rank solution of the SDP relaxation has rank 2;*
3. *The solution matrix has $Y = X^T X$ or $Trace(Y - X^T X) = 0$.*

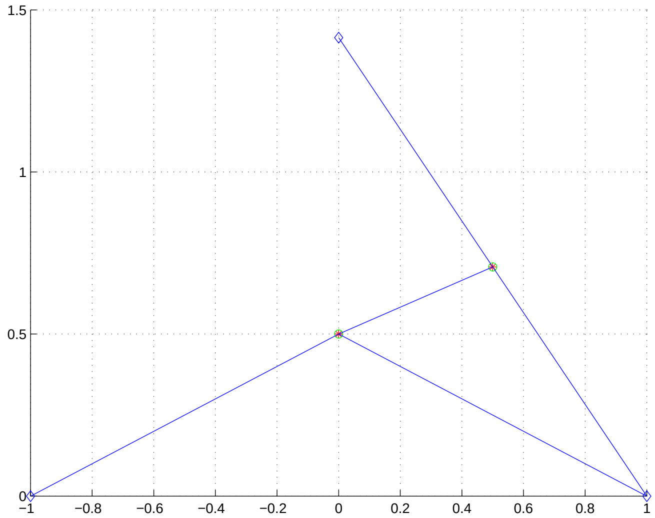
When an optimal dual (stress) slack matrix has rank n, then the problem is 2-strongly-localizable.

If one sensor with its edge lengths to at least three anchors (in general positions) specified, then it is 2-strongly-localizable

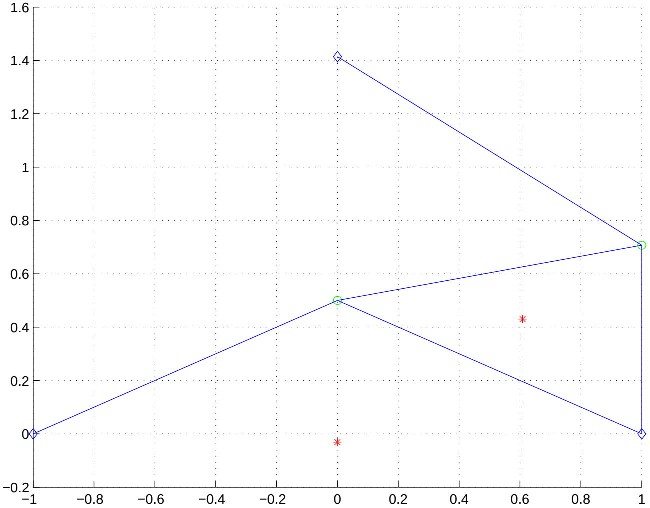
Two sensor-Three anchors: Strongly Localizable



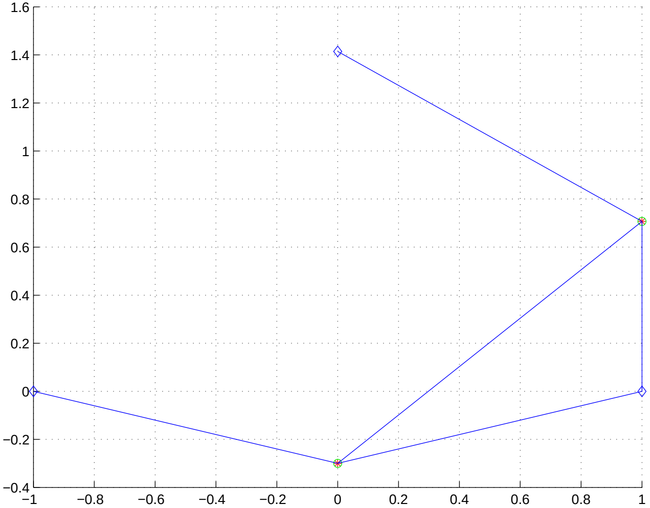
Two sensor-Three anchors: Localizable but not Strongly



Two sensor-Three anchors: Not localizable



Two sensor-Three anchors: Strongly Localizable



Localize All Localizable Points

Theorem

(So and Y 2005) If a problem (graph) contains a subproblem (subgraph) that is localizable, then the submatrix solution corresponding to the subproblem in the SDP solution has rank 2. That is, the SDP relaxation computes a solution that localize all possibly localizable unknown sensor points.

Implication: Diagonals of "co-variance" matrix

$$\bar{Y} - \bar{X}^T \bar{X},$$

$\bar{Y}_{jj} - \|\bar{\mathbf{x}}_j\|^2$, can be used as a measure to see whether jth sensor's estimated position is reliable or not (Biswas and Y 2004).

Anchor Free Localization

Find a rank-d symmetric matrix $Z \in \mathbf{R}^{n\times n)}$ such that

$$\begin{array}{l} (\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_i - \mathbf{e}_j)^T \bullet Z = d_{ij}^2, \ \forall\ i,j \in N_x,\ i < j, \\ Z \succeq \mathbf{0}. \end{array}$$

$$\begin{array}{rl} \text{minimize} & I \bullet Z \\ \text{s.t.} & (\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_i - \mathbf{e}_j)^T \bullet Z = d_{ij}^2, \ \forall\ i,j \in N_x,\ i < j, \\ & Z \succeq \mathbf{0}. \end{array}$$

Theorem

(Biswas et al. 2006) The sensor network is d-uniquely-localizable if and only if the solution of the SDP problem is unique and it has rank d.

Generically Unique Localizability

- The d-localizability depends on graph N_x combinatorics as well as distance measurements d_{ij}.
- Is there a sparse graph that is generically d-localizable, that is, independent of distance measurements?

Trilateration Graphs

A trilaterative ordering in dimension d for a graph G is an ordering of the vertices $1, \cdots, d+1, d+2, \cdots, n$ such that K_{d+1}, the complete graph of the first $d+1$ vertices, is in G, and every vertex $j > d+1$ has $d+1$ edges connected to its preceding vertices on the sequence.

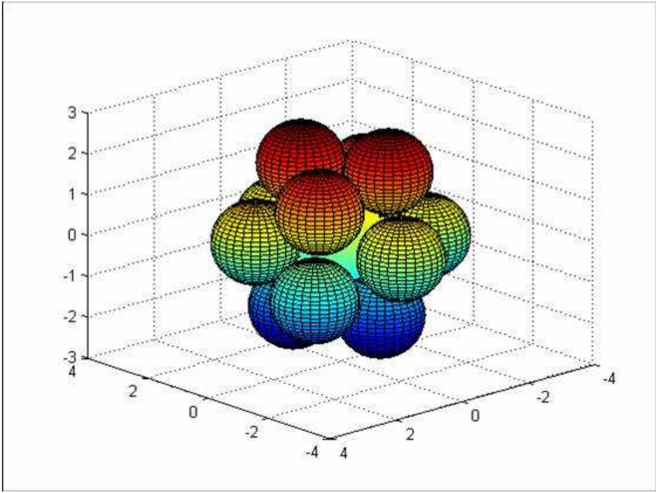
Graphs for which a trilaterative ordering exists in dimension d are called trilateration graphs in dimension d (or d-trilateration graphs). A spanning d-trilateration graph is a d-trilateration and contains every vertex of the graph.

Theorem

(Zhu, So and Y 2009) The spanning trilateration graph in dimension d is generically d-localizable. Moreover, it is a near optimal (with only $O(n)$ edges), in terms of information-theoretical complexity, and generically d-localizable graph.

The Kissing Problem

- Given a unit center sphere, the maximum number of unit spheres, in d dimensions, can touch or kiss the center sphere?
- General Solutions does not exist.
- Delsarte Method uses linear programming to provide an upper bound on the number of spheres.
- K(1)=2, K(2)=6, K(3)= 12, K(8) = 240, K(24) = 196650.
- K(4) = 24: proved using Delsarte Method by Oleg Musin only 3 years ago.
- For other dimensions, lower bounds have been provided by constructing a lattice structure. There also exists a bound using the Riemann zeta function, but is non-constructive.



3
2
1
0
-1
-2
-3
4
2
0
-2
-4
4
2
0
-2
-4

The Kissing Problem as Localization

Given n-balls, find the lowest-rank solution to

$$\begin{aligned} (\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_i - \mathbf{e}_j)^T \bullet Z &\geq 1, \ \forall i < j \leq n, \\ \mathbf{e}_i \mathbf{e}_i^T \bullet Z &= 1, \ \forall i, \\ Z &\succeq \mathbf{0}. \end{aligned}$$

From the Approximate Low-Rank Theorem,

Corollary

One can have n-balls kissed in dimension-$O(\log(n))$ space where the distance error is below any fixed ϵ.

Search for a Low-Rank Solution?

Construct a nonzero SDP objective function to reduce the rank of a solution.

$$\begin{array}{lrl} \min & C \bullet Z & \\ \text{s.t.} & (\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_i - \mathbf{e}_j)^T \bullet Z & \geq 1, \ \forall i < j \leq n, \\ & \mathbf{e}_i \mathbf{e}_i^T \bullet Z & = 1, \ \forall i, \\ & Z & \succeq \mathbf{0}. \end{array}$$

Search for Low-Rank Solution?

The Gegenbauer polynomial:

$$G_0^{(r)}(t) = 1,\ G_1^{(r)}(t) = t,\ \ldots,$$

$$G_k^{(r)}(t) = \frac{(2k + r - 4)tG_{k-1}^{(r)}(t) - (k-1)G_{k-2}^{(r)}(t)}{k + r - 3}.$$

Given symmetric matrix $Y \succeq 0$ with rank r and all its diagonals equal 1, Schoenberg's theorem on the Gegenbauer polynomial:

Theorem

The Gegenbauer polynomial matrix, $[G_k^{(r)}(y_{ij})]$, remains positive semidefinite for $k = 0, \ldots$, where symmetric matrix $[G_k^{(r)}(y_{ij})]$ has the same dimension of Y and its corresponding component equals $G_k^{(r)}(y_{ij})$.