REPORT ON ASYMPTOTIC CONES

SYLVIA CARLISLE AND FRANÇOISE POINT

Say G is a finitely generated group. Let $Y^{(R)}$ be the asymptotic cone of this group as constructed in [3]. We can consider this as a metric structure. The questions we looked at fall under the general theme of how continuous logic and model theory for metric structures might be useful in answering questions about asymptotic cones.

1. Asymptotic cones.

The following outlines the construction in [3]. An asymptotic cone associated with a finitely generated group (G, e), a non-principal ultrafilter U on ω and an infinite natural number $R := [n_i]_U$ is defined as follows.

Let X be a finite set of generators for G and assume it is closed under inverses. Denote by d_X the word metric on G associated with X. First one defines the length of an element g of G as the length of a shortest word in X equal to g. Then $d_X(g, h) =$ length of $g^{-1}h$. One checks that this is a discrete metric on G which is left invariant by multiplication by elements of G (i.e. G embeds in $Isom((G, d_X))$. If one chooses another finite system Y of generators, one obtains a metric space (G, d_Y) which is quasi-isometric to (G, d_X) (the identity map is a quasi-isometry). (For the definition of quasi-isometry, see [6].)

Let $G^{\omega}{}_U := \prod_U (G, e, d_X/n_i)$ be the ultraproduct of the discrete metric spaces $(G, e, d_X/n_i)$. Let $G_U^{\omega\,b} := \{[g_i]_U : \exists r \in {}^{>0}\ d_X(g_i, e) \leq n_i r\}$ be the subset of elements of G_U^{ω} at bounded distance for d/R of the identity. Let η be the set of infinitely small elements namely $\eta := \{[g_i]_U : \forall \epsilon \in {}^{>0}\ d_X(g_i, e) \leq n_i\epsilon\}$. It is a subgroup of G_U^{ω}. Finally we consider $G^{\omega}{}_U{}^b/\eta := (Y_R, d_R, e)$ and we call this space the asymptotic cone associated to (G, e, d_X) and R. Note that the spaces $(Y_R, d_X/R)$ and $(Y_R, d_Y/R)$ are bi-Lipchitz equivalent. S. Thomas and B. Velickovic gave an example of a finitely generated group and two ultrafilters such that the corresponding associated cones are not homeomorphic ([7]). A reference for this paragraph is [6].

Recall that a *geodesic* in a metric space Y between a and b at distance $r \in$ is the image by an isometry f of the interval $[0, r]$ with $f(0) = a$ and $f(r) = b$. A *geodesic space* is a space (Y, d) such that between two points there is a geodesic between them. A *path* γ between a and b in Y is the image by a continuous function g of an interval $[0, 1]$ of with $g(0) = a$ and $g(1) = b$. The *length* of this path $\gamma = g([0, 1])$ is equal the supremum taken over all partitions $0 = t_0 < t_1 < \cdots < t_N = 1$ of $[0, 1]$ of $\sum_{i=0}^{N} d(g(t_i), g(t_{i+1}))$. Let us call an ϵ-geodesic between two points a and b a path of length $\leq d(a, b) + \epsilon$. A *length* space is a (connected) metric space (Y, d) such that between any two points a, b and for every $\epsilon \in {}^{>0}$ there is an ϵ-geodesic between a and b.

Recall that a metric space is proper if its closed balls of finite radius are compact. A locally compact, complete, geodesic space is proper.

Immediate consequences of the above construction of an asymptotic cone are the following: such metric space is:

(1) homogeneous,
(2) geodesic,
(3) complete.

Recall that non-principal ultraproducts over ω are $\aleph_1$-saturated. The first property follows from the fact that the metric is left-invariant. The second property follows from the definition of the word metric and $\aleph_1$-saturation. The last property is a direct consequence of $\aleph_1$-saturation.

These metric spaces were introduced by M. Gromov to prove that a finitely generated group of polynomial growth is nilpotent-by-finite. In that case, he showed that the asymptotic cone was proper, namely that the balls were compact. Moreover that it was finite-dimensional (the dimension being equal to the growth degree). Then in the case the group was not abelian by finite, he showed that there was a non-trivial action by conjugation by an element of η and he used the solution to Hilbert fifth problem to show the existence of a subgroup G_0 of G with $G/G_0 \cong$ (up to finite index quotients) and use induction on the growth degree. (See [4]).

2. Continuous logic

We had many questions about the expressive power of continuous logic in this setting. What language should we work with? What can actually be said in the theory of such a metric structure? (We will abbreviate continuous logic by cclogic.)

We will work in an arbitrary complete metric space (Y, d) with a distinguished point e, sorts for the balls $B(e, n)$ of center e and radius n, $n \in \omega$ and possibly uniformly continuous functions between the sorts. (We will simply denote this space by Y.) When necessary we will restrict ourselves to a smaller class of metric spaces but that will contain in any case the asymptotic cones. For instance in the case of asymptotic cones, the first thing one might try to express is how the group acts on the cone.

The properties we can express in cclogic are the following:

(1) Being a length space
(2) $B(e, 1)$ is totally bounded i.e. $\forall \epsilon > 0$ there exists a finite ϵ-net.
(3) $B(e, 1)$ has Minkowski dimension $\leq s$,
(4) Being an homogeneous space,
(5) Being an δ-hyperbolic space.

The ball $B(e, 1)$ is covered by n_ϵ balls of radius ϵ can be said by

$$inf_{x_1} \cdots inf_{x_{n_\epsilon}} sup_y min_{i=1,\cdots,n_\epsilon} d(x_i, y) \dot{-} \epsilon = 0.$$

2.1. **Lemma.** *Y is a length space iff Y satisfies the following sentence:*

$$sup_x sup_y inf_z (max\{|d(x, z) \dot{-} 1/2d(x, y)|, |d(y, z) \dot{-} 1/2d(x, y)|\} = 0).$$

Proof. We use the fact that being a geodesic space is equivalent to the *mid-point* property, namely that given any two points a, b, there exists c such that $d(a, c) + d(c, b) = d(a, b)$. We cannot quite express this property in our logic. However its approximation gives us the length space property. (See [1] page 32.) □

2.2. **Lemma.** *An $\aleph_1$-saturated metric space (Y, d) which is a length space is a geodesic space.* □

2.3. **Corollary.** *An asymptotic cone is a geodesic space iff it satisfies the above sentence.*

2.4. **Lemma.** *Let Y be an homogeneous space. Then, the fact that the ball $B(e,1)$ has Minkowski dimension $\leq s$ can be expressed in cclogic.*

Proof. Y has Minkowski dimension $\leq s$ iff
$sup_{\epsilon>0} f(\epsilon)(\epsilon)^s < \infty$, where $f(\epsilon)$ is the cardinality of a covering of $B(e,1)$ by balls of radius ϵ, iff
$\exists c > 0 \forall \epsilon > 0\ f(\epsilon)(\epsilon)^s < c$ iff
$\exists x_1 \cdots \exists x_{f(\epsilon)} \forall y_1 \cdots \forall y_{f(\epsilon)} \forall z\ [max_i\{d(x_i,e) \leq 1\}]\ \&\ [max_i d(x_i,y_i) \leq \epsilon \rightarrow \sum_{i=1}^{f(\epsilon)} d(x_i,y_i)^s]\ \&$
$[d(z,e) \leq 1 \rightarrow min_i d(z,x_i) \leq \epsilon]$ iff
letting $\epsilon = 1/k;\ k \in \omega$
$inf_{x_i} sup_{y_j} sup_z\ max\{max_i d(x_i,e) \dot{-} 1, \sum_i (d(x_i,y_i)^s \dot{-} c) \dot{-} (max_i d(x_i,y_i) \dot{-} \epsilon),$
$(min_i d(z,x_i) \dot{-} \epsilon) \dot{-} (d(z,e) \dot{-} 1) = 0\}$. □

Recall that the asymptotic cones associated to the finitely generated free groups are 0-hyperbolic spaces and more generally those associated to δ-hyperbolic groups are 0-hyperbolic spaces. Also, a geodesic space which is 0-hyperbolic embeds in an -tree. [An -tree is a uniquely geodesic space, where if two geodesic segments meet only at a common end point, their union is a geodesic segment.] (See [6].)

2.5. **Lemma.** *Being δ-hyperbolic can be expressed in cclogic.*

Proof. One uses the following characterization of δ-hyperbolic spaces (see Proposition 1.6 [2]): $d(x,y) + d(z,w) \leq max\{d(x,z) + d(y,w), d(x,w) + d(y,z)\} + 2\delta$.
$(Y, \bigcup_n B(e,n)) \models sup_{x,y,z,w}((d(x,y) \dot{+} d(z,w)) \dot{-} max\{d(x,z) \dot{+} d(y,w), d(x,w) \dot{+} d(y,z)\} \dot{-} 2\delta = 0$. □

3. D-Definable sets

Recall that an d-definable set is a set D such that $d(x,D)$ can be expressed by a definable predicate.

3.1. **Lemma.** *Let (Y,d) be a geodesic space. Then, the closed ball of center e and radius r, $\bar{B} := \bar{B}(e,r)$, is a d-definable subset.*

Let $\phi(x,y) := d(x,y) \dot{+} (d(y,e) \dot{-} r)$. Note that $\phi(x,y) \geq d(x,B)$ if $d(y,e) \geq r$ and it is equal to $d(x,y)$ whenever $d(y,e) \leq r$.

Then, we have that $d(x,B) = inf_y \phi(x,y)$. Indeed, if $x \in B$, then take $x = y$.

If $x \notin B$, and $y \in B$ then $\phi(x, y) = d(x, y)$. Let $y \notin B$. We have that $\phi(x, y) = d(x, y) + (d(y, e) - r)$. Since $y \notin B$, then there is a geodesic from e to y. Since $d(y, e) > r$, there is a point on that geodesic at distance r from e, call it y_0 and $d(y, e) - r = d(y, y_0) + d(y_0, e) - r = d(y, y_0)$. Now, $\phi(x, y) = d(x, y) + d(y, y_0) \geq d(x, B)$. □

Note that our proof above relies on the fact that Y is a geodesic space, and such a ball should be d-definable in any geodesic space. Also, the same formula should work in a length space. As similar thing works to show the ball centered at any point, is also d-definable.

4. Questions

(1) Suppose (Y_1, d_1) and (Y_2, d_2) are cc-elementary equivalent, then are they quasi-isometric? Vice-versa?

(2) Given a finitely generated group, what can we say about the theory of the class of all its asymptotic cones (in first-order logic or cclogic)?

Given two finitely generated groups which are elementary equivalent, are their asymptotic cones, with respect to the same ultrafilter and non-standard natural number R, bi-Lipschitz equivalent?

(3) Is the class of complete metric spaces coming from asymptotic cones cc axiomatizable?

(4) Can we express that an asymptotic cone is coming from an abelian group versus a nilpotent non abelian group? (In the latter case, the topological dimension of the space is not the same as its Minkowski dimension.) (See [8]).

For instance, can one distinguish the asymptotic cone of the Heisenberg group from that of ^{4} by looking at their continuous theories ([9])?

(5) Can a δ-hyperbolic space, $\delta > 0$, be an asymptotic cone associated with a finitely generated group?

(6) Is being simply connected a cclogic property?

(7) Given two points a, b is the set of geodesics between these two points a d-definable set?

(8) Can we express *curvature* in cclogic?

(9) Can having finite asymptotic dimension ([6] section 9) be expressed in cc logic?

(10) Since we can form the asymptotic cone of any metric space, how might these ideas be useful in a more general setting. For example, can one learn something by taking the asymptotic cone of a first order structure with the discrete metric?

5. Further thoughts.

5.1. **Question.** *Is the complement of a ball also a d-definable set?*

Denote by B^c the closure of the complement of the ball $B = B(e, r)$.

The answer is yes, if given a geodesic it can be prolonged to a geodesic of length $\geq r$. So, given an element $y \in B(e, r)$, there would exist a geodesic from e passing through y, of length $\geq r$. Then on this geodesic, we can always find an element at distance r from e. Therefore, $d(y_0, B^c) \leq r - d(y_0, e)$.

Suppose that for all $y_0 \in B$ we have $d(y_0, B^c) \leq r - d(y_0, e)$ (*), then for $y \in B$, we have that $d(y, B^c) \leq inf_{z \in B}\{d(y, z) + (r - d(z, e))\}$.

Define $\phi^c(x, y) := d(x, y) \dot{+} (r \dot{-} d(y, e))$. Then if $y \notin B$, then $\phi^c(x, y) = d(x, y)$. Suppose $x, y \in B$, we have to show that $\phi^c(x, y) \geq d(x, B^c)$.

A Cayley graph has the *extension property for geodesics segments* if for any pair of vertices x, y there exists a vertex z such that $d(x, z) = d(x, y)+1$. The Cayley graph of a finitely presented group $< S/R >$ satisfying the small cancellation hypothesis $C'(1/6)$ has this property ([5] p.83)

So, the asymptotic cone associated with a group with this property has also the prolongation property for geodesics.

Measure.

Note that the asymptotic cone associated with a finitely generated group (G, X) can be endowed with a Keisler measure μ defined as follows.

Let $\phi(x)$ be an $\mathcal{L}$-formula, where $\mathcal{L}$ is the language of groups. Let $D(G) := \{g \in G : G \models \phi(g)\}$ and $D := \{[g_i]_U \in G^\omega/U : G \models \phi(g_i)\}$. Define $\mu(D.\eta) = st([|\{g \in D(G) : \ d_X(g, e) \leq n\}|/b(n)]_U)$, where st denotes the standard part map and $b(n)$ the cardinality of the ball of center e and radius n. Note that we may allow parameters from G. It is a finitely additive measure. It is left invariant whenever $\mu[((D \triangle a.D) \cap B(e,n))]$ goes to 0 whenever n goes to infinity.

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