REPORT ON NON-COMMUTATIVE PROBABILITIES AND VON NEUMANN ALGEBRAS

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We sought to understand non-commutative probabilities from a model-theoretic point of view via the study of von Neumann algebras equipped with a finite trace.

We formalised the unit balls of such structures as continuous structures and showed they formed an elementary class. The theory $T_0$ axiomatising this class can be taken to be universal if we add a unary function symbol $\_2$ such that if $\|a\| \leq \frac{1}{2}$ then $2a = 2a$.

The following sets were shown to be definable:

(1) The set of all self-adjoint elements.

(2) The set of all projections.

(3) The set of all projections below a given projection $p$ (uniformly in $p$).

(4) The set of all projections such that $\tau(p) = \frac{1}{2}$.

Using these facts we can explicitly write an axiom $NA$ saying that the von Neumann algebra is atomless, and modulo that, an axiom $NC$ saying it is centreless (i.e., a $II_1$ factor). Furthermore, one also shows the sets above are in fact indefinite, from which it follows that the new axioms $NA$ and $NC$ are $\forall \exists$. Thus the theory $T_1 := T_0 + NA + NC$, whose models are precisely the $II_1$ factors, is an $\forall \exists$-theory. As $T_0$ and $T_1$ are companions (every model of one embeds in one of the other), we conclude that every existentially closed (e.c.) von Neumann algebra (with a finite trace) is a $II_1$ factor.

We can further show that the class of von Neumann algebras which embed in some ultrapower of $R$ is elementary, axiomatised by a universal theory $T_0^e$.

0.1. Question. Find explicit axioms for $T_0^e$.

This is a universal theory, although we still have to find an explicit statement of the embeddability axiom. We know that $T_0^e$ and $T_1^e := T_1 \cup T_0^e$ are companions, so every e.c. embeddable von Neumann algebra is an embeddable $II_1$ factor.
0.2. **Question.** Does $T^c_1$ admit quantifier elimination?

A reasonable plan of approach would be to show that $R^\mathcal{U}$ admits quantifier-free back-and-forth. There are indications this may be true, but more work is required.

Since $R^\mathcal{U}$ is $\aleph_1$-saturated, this means that $\text{Th}(R^\mathcal{U}) = \text{Th}(R)$ eliminates quantifiers. This theory clearly contains $T^c_1$. Are they equal?

0.3. **Question.** Do e.c. models of $T^c_1$ admit a “nice” notion of independence?

The theory $T^c_1$ has the independence property, and is therefore unstable. There is still a vague hope it might be simple.

There exists many possible notions of independence in von Neumann algebras, the weakest of which is the following:

$$A \perp_B C \iff (\forall a \in \langle AC \rangle) (\forall b \in \langle BC \rangle) \left( \langle C \rangle(ab) = \langle C \rangle(a) \langle C \rangle(b) \right)$$

$$\iff (\forall a \in \langle AC \rangle) \left( \langle C \rangle(a) = \langle BC \rangle(a) \right).$$

Here $\mathcal{B}$ denotes conditional expectation with respect to $\mathcal{B}$, i.e., orthogonal projection in the sense of the inner product $\langle x, y \rangle = \tau(x^*y)$.

We verified this notion of independence satisfies all axioms of a stable/simple notion of independence except for stationarity/independence theorem. Indeed, it seems that with stronger notions of independence we may run into trouble with the local character. While stationarity is known to fail (by direct constructions, as well as from the fact that the theory is unstable), there may still be hope the independence theorem holds. If that is the case then the theory is simple with $\perp$ coinciding with non-dividing.