REPORT ON BANACH SPACES

JOSÉ IOVINO, WILLIAM JOHNSON, CARLOS ORTIZ, IVES RAYNAUD,
CHRISTIAN ROSENDAL, ALEX USVYATSOV, AND MONICA VANDIEREN

This group brought together researchers with backgrounds in functional analysis (Johnson, Raynaud), set theory (Rosendal), and model theory (Iovino, Ortiz, Usvyatsov, and VanDieren). The discussions were motivated by the question of what model theoretic conditions on a separable, infinite dimensional Banach space X ensure that every infinite dimensional subspace of X contains one of the classical sequence spaces (ℓ_p and c_0).

A well-known result of Krivine and Maurey (1979) states that for every stable Banach X space there exists $p \in [1, \infty)$ such that X contains ℓ_p almost isometrically; stable, in this context, means that whenever (a_m) and (b_n) are bounded sequences in X and $\mathcal{U}, \mathcal{V}$ are ultrafilters on , one has

$$\lim_{m,\mathcal{U}} \lim_{n,\mathcal{V}} \|a_m + b_n\| = \lim_{n,\mathcal{V}} \lim_{m,\mathcal{U}} \|a_m + b_n\|.$$

From a model theoretic point of view, this means that no quantifier-free formula $\psi(x, y)$, where x and y are finite tuples, defines an order in X.

We wished to see if a similar conclusion could be proved under weaker hypotheses.

A Banach space X is stable in this sense if and only if there exists a binary operation $*$, called *convolution*, on the space $\mathcal{T}(X)$ of quantifier-free 1-types over X (regarded as a topological space with the product topology) such that:

(1) $*$ extends the addition of X to $\mathcal{T}(X)$,
(2) $*$ is separately continuous,
(3) $*$ is commutative.

These properties play a crucial role in the argument of Krivine and Maurey.

From the point of view of model theory, the convolution $*$ is defined as follows: if $\sigma, \tau \in \mathcal{T}(X)$, $\sigma * \tau$ is $\operatorname{tp}(a + b/X)$ where a, b are realizations of σ, τ, respectively, that are independent over X.

We observed that without the presence of stability, it is possible to define a similar convolution, not on types over X, but on what Maurey, in an unpublished manuscript, called *strong types*. Intuitively, strong types are types over $\mathcal{T}(X)$ which are finitely realized in X. Strong types can be viewed from a model-theoretic perspective as follows: one fixes a set $A \supset X$ where every n-type over X is realized, and defines a strong type for X over A as a type over A which is finitely realized over X.

Such types are useful in classification theory. The reason is that given strong types p, q for X over A and a realization a of p, there exists a unique extension q' of q over $A \cup \{a\}$ such that q' is a strong type for X. In analogy with the stable context, if b is a realization of q', one could say that b is "independent" of a over X. One can then define $p*q$ as the strong type $\mathrm{tp}(a+b/A)$; this convolution is well defined, in the sense that the definition of $p*q$ does not depend on the particular choice of a and b. The weak concept of independence just described has been used by Shelah for the construction of the kinds of sequences known as *Shelah sequences*; such sequences function as analogs of Morley sequences in unstable contexts. Shelah sequences exist for any type, but for strong types, they are uniquely determined by the type. For quantifier-free strong types, the convolution defined above corresponds precisely to the convolution defined by Maurey in the manuscript where he introduced the definition of strong type. Shelah sequences correspond to spreading models of X.(Morley sequences correspond to spreading models of stable Banach spaces.)

It is then natural to try to replicate the Krivine-Maurey argument in unstable contexts by using strong types in place of types, and the strong type convolution in place the stable convolution of Krivine and Maurey. The main obstacle is that, without the stability assumption, the convolution need not be commutative; in other words, the weak concept of independence defined above does not satisfy the symmetry property. It was noted during the discussions that if p, q are strong types for X and least one of p and q is definable, then $p * q = q * p$.

It was also noted that to carry out the argument of Krivine and Maurey one does not need commutativity of $*$ on all of $\mathcal{T}(X)$. Suppose that X is stable and $\tau \in \mathcal{T}(X)$ is such that none of the types of the form

$$(1) \qquad \lambda_1\tau * \cdots * \lambda_n\tau, \qquad (\lambda_1, \ldots, \lambda_n \text{ are scalars and } n < \omega)$$

is trivial; then the Krivine-Maurey argument can be carried out if $*$ is commutative for types in the closure of $[\tau]$ in $\mathcal{T}(X)$, where $[\tau]$ denotes the set of all types of the form (1). The question arose of what this condition translates into, in terms of the geometry of X, if types are replaced by strong types.

The final sessions focused on understanding the concept of definability over X, both for types over X and strong types, and its connection with Krivine-Maurey stability and uniqueness of Shelah sequences.